京都大学
KYOTO UNIVERSITY

Université de Lille

Research Institute for Mathematical Sciences - Kyoto University, Japan

PROMENADE IN INTER-UNIVERSAL TEICHMÜLLER THEORY - 復元

Online Seminar - Algebraic & Arithmetic Geometry

Laboratoire Paul Painlevé - Université de Lille, France

INTRODUCTION

Mochizuki's Inter-Universal Teichmüller theory (IUT) has been developed with the goal of providing a deeper understanding of the *abc-conjecture*– for every $\varepsilon > 0$, the relation $\mathrm{rad}(abc)^{1+\varepsilon} < c$ has only finitely many solutions for coprime integers $(a, b, c) \in \mathbb{N}_{>0}$ such that $a + b = c$ (abc). This conjecture, that is deeply rooted in arithmetic geometry, is to be understood as *a certain rigidity property on the intertwining of the multiplicative and additive monoid structures of* $(\mathbb{N}_{>0}, \boxplus, \boxtimes)$, whose estimate is hidden within the many isomorphic identifications used in arithmetic and Diophantine geometry.

The seminal achievement of Mochizuki's IUT is to provide *a new geometry* that brings an estimate of the (abc) rigidity property within reach. IUT theory is a geometry of the moduli stack of elliptic curves $\mathcal{M}_{1,1}$ whose K-points are endowed with certain *rigid Diophantine* arithmetic line bundle invariants with place-wise compatible arithmetic and geometric symmetries, and that are embedded in various types of *non-rigid anabelian étale containers*. The consideration of rings/schemes and fields in terms of mono-anabelian geometry and up-to some indeterminacies, for example at nonarchimedean places of Ks, as *abstract* $\boxplus/\boxtimes$*-monoids*, allows the decoupling of their $\boxplus/\boxtimes$-monoids structures. This deconstruction-reconstruction process – or *Fukugen* 復元 , relies on the non (mono) anabelianity of sub-p-adic fields and allows to track the isomorphic identifications of ring structures, which in turns provides an abc-estimate at the level of heights of line bundles.

This theory relies on a 20-year work of Mochizuki in anabelian geometry, Hodge-Arakelov and p-adic Teichmüller theories. The various aspects involved – mono-anabelian transport of rigid properties, Diophantine invariants in anabelian geometry, categorical constructions, structures deformations – will appeal to as many arithmetic geometers.

The goal of this seminar is to give access to non-experts to techniques and insights of IUT by starting with the illustration of seminal principles and theories on which relies the work of Mochizuki. It is our hope that this approach will provide the participants **(1)** with the necessary keys for the practice of IUT theory, **(2)** with a guide towards the appreciation of Mochizuki's proof of the abc-conjecture, and **(3)** for further ramifications of IUT within their own field of research.

Acknowledgement. The organizer BC would like to express his gratitude to *Shinichi Mochizuki* for regular meetings and discussions during which he could properly be introduced to the insights, techniques and future developments of IUT. He also thanks *Emmanuel Lepage and Arata Minamide* for sharing their understanding of the theory, and *Yuichiro Hoshi and Shota Tsujimura* whose corrections led to a finer version of these notes. The first organizer benefited from the support of *Akio Tamagawa* and the hospitality of RIMS Kyoto University during the preparation of this seminar.

"Progress in mathematics is [...] a *much more complicated family tree* [...], a *much more complicated organism*, [...] whose growth is sustained by an intricate mechanism of interaction among a vast multitude of branches, some of which sprout not from branches of relatively recent vintage, but rather from much older, more ancestral branches of the organism that were entirely irrelevant to the recent growth of the organism."

S. Mochizuki (2011)

As a model for culture, *the rhizome* resists the organizational structure of the root-tree system which charts causality along chronological lines [...]; A rhizome is characterized by "ceaselessly established connections between semiotic chains, organizations of power, and circumstances relative to the arts, sciences, and social struggles." *The rhizome presents history and culture as a map or wide array of attractions and influences with no specific origin or genesis.*

From G. Deleuze, F. Guittari (1983)

Contents

※ *In order to keep the length of this guide (incl.* ~ 25 *tables, figures, and diagrams) strictly shorter than the IUT corpus –* ~ 1200 *pages with a piece of anabelian geometry,* ~ 675 *pages for the canon, and* ~ 170 *pages for the introductory [Alien] – some details have been omitted, some approximations were made; they should be negligible for our goal. Content will be updated according to the progress of the seminar, see version and date.*

Programme

The following programme is intended for *non-experts and young researchers in arithmetic geometry*, with the goal to serve as a guide towards a general understanding of results, insights and techniques of Inter-Universal Teichmüller theory. The organization around three topics – Diophantine Geometry, Inter-Universal Geometry, and Anabelian Geometry – emphasizes the grounding of IUT into classical arithmetic-geometry theories, which in return serve as many bridges towards IUT's new insights.

Speakers & Talks. Each speaker will freely determine the balance between expository and technicity, decide on which material to develop – from elementary to advanced topics, and whether or not to focus on complements. At least 30 minutes should be spent for *contextualizing the talk* with respect to the programme; A specific effort will be given **(1)** on a rigorous presentation of elementary notions, and **(2)** on the use of examples and diagrams to support the presentation.

Speakers should feel free to contact the organizers for informal discussions during their preparation and for access to (additional) references. The Talk numbering is no indication of any chronological ordering, except for Talk 0 that will give a general overview of each topic – see §Talks and Speakers.

Modus Operandi & Leitfaden. As a new geometry, the essence of Mochizuki's IUT is to introduce a new semiotic system – formalism, terminology, and their interactions – that can be unsettling at first. This programme proposes a 3 layers approach with precise references, examples, and analogies.

Because IUT discovery also benefits from a non-linear and spiralling approach, we provide further indications for an independent wandering: Mochizuki recommends to start with the introductory [Alien] – young arithmetic-geometers can also consult [Fes15] for a shorter overview. We also recommend to begin with *§Intro - §3.6-7 ibid.* for a direct encounter with IUT's semiotic, then to follow one's own topics of interest according to Fig. 1, which also indicates some topic-wise references as entry-points – [EtTh], [GenEll], etc. Within the "canon" [IUTChI]-[IUTChIV], our recommendation is to start with [IUTChIII] §Introduction. Intuition of the reader can further rely on the strongly consistent terminology of IUT – e.g. Frobenioid, mono-anabelian transport, arithmetic analytic.

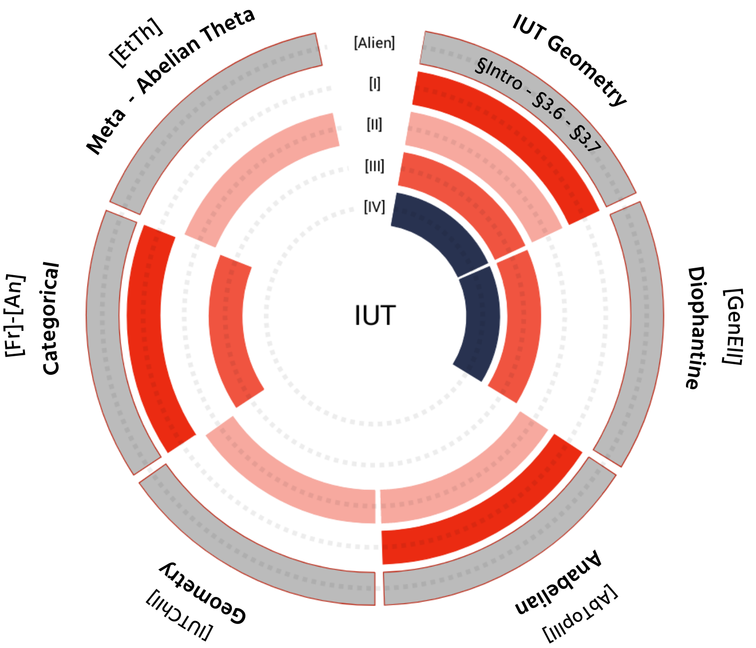


Fig. 1. *IUT, Topics & References as potential entry points.*

- ***Diophantine:*** *Heights, Faltings' isogenies & Abc.*
- ***Anabelian:*** *Mono-anabelian reconstruction & Tripodal transports.*
- ***Geometry:*** *Multiradiality, Coricity & Arithmetic Analyticity vs Holomorphicity.*
- ***Categorical:*** *Frobenioids, Anabelioid, Prime Strips & Hodge Theaters.*
- ***Meta-Abelian Theta:*** *Mumford's abelian constructions & Kummer theory.*

※ *We have also found the synthetic and self-content [Yam17] to be particularly helpful as a bridge between [Alien] and the "canon".*

※ *Hodge-Arakelov and p-adic Teichmüller theories stand as important models for IUT, which also relies on key categorical constructions – e.g. Frobenioids and anabelioids. These aspects* ***are not included*** *in this programme – we refer to [Alien] and the canon for references – they can be the object of additional talks by specialists.*

Inter-Universal Teichmüller theory (1991 - 2012 - 2020 - . . .) One needs to recall that IUT is the abutment of a 20-years research programme – involving p-adic Teichmüller, Hodge-Arakelov, and absolute/combinatorial/mono-anabelian geometry theories – that Mochizuki started during his PhD in Princeton **1991**. After a 6-years private seminar organized by Sh. Mochizuki and F. Kato (July 2005 - March 2011)), the "canon" of the theory was publicly released in **2012** – see [Fes15] §1.4 and §3.1 – to be finally accepted for publication in *Kyoto's Publication of the Research Institute for Mathematical Sciences* on April 4, **2020**.

In between, IUT has been learned and used inside and, independently, outside the Japanese Mathematics community. As a new geometry, it gives access to new invariants, symmetries and principles, that will certainly be recognized and thus appeal to both Diophantine and anabelian geometers – among others: **(1)** the symmetric gluing of prime and geometric cusps of Talk 2.1, **(2)** the arithmetic properties of Galois representations with big image in relation with Talk 1.3, and **(3)** the embedding of Frobenius-like (local) objects in (global) étale-like and Galois-deformable ones of Talk 2.3.

The now very active international community of IUT geometers will meet for a special 2021 "Expanding Horizons of Inter-universal Teichmüller theory" semester in Kyoto, where the latest results surrounding IUT (anabelian geometry, Grothendieck-Teichmüller theory, abc-type) will be announced and discussed.

※ *Professor Mochizuki will give an IUT talk at the Berkeley Mathematical Colloquium by Zoom on Thursday Nov. 5th, 2020 at 4:10pm to 5:00pm (PST).*

Introductory References.

[Fes15] I. Fesenko, "Arithmetic deformation theory via arithmetic fundamental groups and nonarchimedean theta-functions, notes on the work of Shinichi Mochizuki," *Eur. J. Math.*, vol. 1, no. 3, pp. 405–440, 2015, MR3401899.

[Hos19] Y. Hoshi, "On the examination and further development of inter-universal Teichmüller theory," *RIMS Kokyuroku Bessatsu*, vol. B76, pp. 79–183, Aug. 2019, In Japanese, Eprint available on-line.

[Alien] S. Mochizuki, "The mathematics of mutually alien copies: From Gaussian integrals to Inter-universal Teichmüller theory," *RIMS Preprint **no. 1854***, 169p. Jul. 2016, Eprint available on-line.

[Yam17] G. Yamashita, "A proof of abc conjecture after Mochizuki," p. 294, Aug. 2017, v6, Eprint available on-line.

Main References.

[GenEll] S. Mochizuki, "Arithmetic elliptic curves in general position," *Math. J. Okayama Univ.*, vol. 52, pp. 1–28, 2010, MR2589843, Eprint available on-line.

[IUTChI] ——, "Inter-universal Teichmüller theory I: Construction of Hodge theaters," *RIMS Preprint **no. 1756***, 186p. Aug. 2012, Eprint available on-line.

[IUTChII] ——, "Inter-universal Teichmüller theory II: Hodge-Arakelov-theoretic evaluation," *RIMS Preprint **no. 1757***, 174p. Aug. 2012, Eprint available on-line.

[IUTChIII] ——, "Inter-universal Teichmüller theory III: Canonical splittings of the log-theta-lattice," *RIMS Preprint **no. 1758***, 199p. Aug. 2012, Eprint available on-line.

[IUTChIV] ——, "Inter-universal Teichmüller theory IV: Log-volume computations and set-theoretic foundations," *RIMS Preprint **no. 1759***, 87p. 2012, Eprint available on-line.

The goal of this section is to translate Masser-Oesterlé's abc conjecture over $\mathbb{N}_{\geqslant 0}$ to Mochizuki's Vojta Generalized conjecture in terms of Faltings' height over the moduli spaces of elliptic curves $\mathcal{M}_{1,1}$. We first relate abc to 2 arithmetic results in terms of Diophantine approximation (Roth's Theorem) and reduction of elliptic curves (Szpiro Conjecture) in Talk 1.1, then recall how to obtain further connection to algebraic geometry via Vojta Conjecture in terms of *bounding heights at points of curves over number fields* [Voj98] in Talk 1.2. We conclude in Talk 1.3 with Mochizuki's approach in terms of *Faltings' invariance of heights under isogenies* [GenEll], which motivates two key points of IUT: **(1)** the construction of a global multiplicative subspace of torsion points of elliptic curve (GMSCG), and **(2)** the decoupling of the ⊞/⊠-monoid structures of ring structures to obtain a global Frobenius-type morphism – see also Talk 2.1.

※ *We refer to [BG06] for elementary definitions of Diophantine Geometry in relation with abc (height, Belyi, etc.) and to [Fes15] §1.2 for an overview. See also [Sil94] Chap. V for Tate's theory of q-curves over p-adic fields.*

TALK 1.1 - ABC & SZPIRO CONJECTURES, ROTH'S THEOREM AND BELYI. An elementary geometrization of abc (strong form) is already related to two results of arithmetic-geometry of two distinct types: **(1)** Roth's Theorem over $\mathbb{Q}$ which is an approximation result of algebraic numbers *à la* Liouville-Thue [BG06] Th. 6.2.3, and **(2)** the generalized Szpiro conjecture on reduction of elliptic curve ibid. Conj. 12.5.11.

Let us recall the abc (strong form), see ibid. Conj. 12.2.2.

Abc Conjecture (Strg.) *For every $\varepsilon > 0$, there exists C_ε such that any coprime triplet $(a, b, c) \in \mathbb{N}_{>0}$ with $a + b = c$ satisfies:*

$$c \leqslant C_\varepsilon \; rad(abc)^{1+\varepsilon}.$$

We refer to ibid. Chap. 12.2 for further details and examples (computations, Fermat curves, etc.)

§Abc-Roth: Diophantine Approximation. It results from a very educative application of Belyi's Lemma – the factorization of a curves morphism by a $\{0, 1, \infty\}$-ramified cover of $\mathbb{P}^1_K$, see ibid. Lem. 12.2.7 – that the strong abc conjecture implies Roth's Theorem, see ibid Th. 12.2.9.

Roth Theorem. *Let $\alpha \in \mathbb{Q}$ be an algebraic number. For $\kappa > 2$, there are only finitely many $\beta \in \mathbb{Q}$ such that:*

$$|\beta - \alpha|_\infty \leqslant 1/H(\beta)^\kappa$$

where $H(\beta) = e^{h(\beta)}$ is the absolute exponential height, and $|-|_\infty$ denotes the usual Archimedean absolute value in $\mathbb{R}$.

Szpiro Conjecture (Gen.) *Let E be an elliptic curve over $\mathbb{Q}$. Then for $\varepsilon > 0$:*

$$\max(|\Delta|, |c_4|^3) \ll_\varepsilon cond(E)^{6+\varepsilon}$$

where c_4 is taken in the minimal Weierstrass equation of E over $\mathbb{Z}$, and Δ denote its global minimal discriminant.

§Abc-Szpiro: Reduction for Elliptic Curves. Put roughly, the conductor $\mathrm{Cond}(E)$ is an ideal that encodes the (good/multiplicative/additive) reduction property of E/K at various $\mathfrak{p} \in \mathrm{Spm}\,\mathcal{O}_K$ – see ibid. §12.5.5 - §12.5.9 for definitions, examples, and references. Some direct estimates via the use of Frey curves (Ex. 12.5.10) establish the equivalence between abc (strong form) and the Szpiro conjectures, see ibib. Th. 12.2.12.

※ *In his full form Belyi's Theorem is a fundamental result of arithmetic-geometry, see ibid. Chap. 12.3 and also Talk 3.3.*

TALK 1.2 - ABC & VOJTA CONJECTURES: HEIGHTS AND RAMIFICATION. Vojta Conjecture in its "generalized" form [Voj98] introduces further elements of arithmetic-geometry in terms of divisors, curves, and number fields. The Diophantine ingredient is here given by Weil's notion of height, see [BG06] §2.4. As a result, one obtains a first connection with abc and the coarse moduli scheme $M_{1,1}$ of one-pointed elliptic curves endowed with $D = [0] + [1] + [\infty]$.

The Vojta conjecture with ramification for curves – see ibid. Conj 14.4.13 & 14.4.10 – is the equivalent form of abc (Strg.) in its number field version as formulated in Conj. 14.4.12.

> **Vojta Conjecture (Curve NF.)** *For all curves C over any number field K, considering $S \leqslant M_K$ a finite set of places on K, D a reduced effective divisor and H a ample line bundle on C, let $\varepsilon > 0$, then*
>
> $$m_{S,D}(P) + h_{K_C}(P) \leqslant d(P) + \varepsilon h_H(P) + O_{[K(P):K]}(1)$$
>
> *holds for every $P \in C \smallsetminus supp(D)$.*

Here, $m_{S,D}$ denotes the proximity function of local heights with respect to D and S of §14.3.1, and $h_\bullet$ denotes the height function with respect to a line bundle.

The Vojta Conjecture for curves over number fields with ramification is proven to be equivalent to the strong abc-conjecture – see ibid. Th. 14.4.16.

※ *A general Vojta for X a general irreducible smooth projective variety can be found in Conj. 14.3.2 that indeed holds for a linear situation in $X = \mathbb{P}^n_K$ – see Th. 14.3.4. In the context of algebraic approximation, this idea of ramified covers also leads to a Roth's Theorem, Th. 14.2.6, which is proven to be strictly equivalent to the original one – see Prop. 14.2.7.*

TALK 1.3 - FROM VOJTA TO MOCHIZUKI: MODULI SPACES OF ELLIPTIC CURVES. The final step toward the arithmetic-geometrization of abc with respect to the moduli space of elliptic curves $M_{1,1}$ is given by Mochizuki in [GenEll] with the following formulation:

> **Vojta Conjecture (Gen.)** *Let C be a smooth proper geom. connected curve over K number field and D a reduced effective divisor such that $C \smallsetminus D$ is hyperbolic. Then $\forall n \in \mathbb{N}_{\geqslant 0}$, $\forall \varepsilon > 0$, $\exists c$ constant such that:*
>
> $$\mathrm{ht}_{\omega_C(D)}(x) \leqslant c + (1+\varepsilon)(\log - \mathrm{diff}_C(x) + \log - \mathrm{cond}_D(x))$$
>
> *for all $x \in (C \smallsetminus D)(K')$ with K' any number fields $[K' : K] \leqslant n$.*

which is proven to reduce to the case $(C = \mathbb{P}^1, D = [0]+[1]+[\infty], K = \mathbb{Q})$ – aka the coarse moduli scheme $M_{1,1}$, see ibid. Th. 2.1. Here ω_C denotes the canonical sheaf on C, $\log - \mathrm{diff}_C(x)$ the log-difference attached to the minimal field of definition of $x \in C(\bar{K})$, and $\log - \mathrm{cond}_D(x)$ the log-conductor similarly attached to D – see [GenEll] Def. 1.5

※ *This new formulation can be seen as the abutment of the Belyi techniques of the Diophatine approximations and of the elliptic reduction constraints of the Szpiro conjecture.*

§Faltings: Towards Anabelian Geometry. The consideration of *Falting's height* $\mathrm{ht}^{\mathrm{Falt}}(-)$ – defined on anabelian variety with respect to metrized line bundles – provide a height theory that is invariant under isogeny (or finite étale morphism) and defined in terms of differential forms – see [Del84] and [GenEll] below 3.1. This can be seen as a final step towards the étale and p-adic techniques of anabelian geometry – see Topic 3.
On the other hand, recall that an arithmetic divisor D first defines an arithmetic line bundle $\mathcal{L} = \mathcal{O}_E(D)$, then *a height function* $\mathrm{ht}_{\mathcal{L}}(\bullet) = deg^{ar}(\bar{\mathcal{L}}_{|\bullet})$ over $E(\bar{\mathbb{Q}})$ via the arithmetic degree – see [GenEll] §1. For $E/\operatorname{Spec}\mathcal{O}_F$, or $\operatorname{Spec}\mathcal{O}_F \to \mathcal{M}_{1,1}$, this height is then equal to Faltings' up to a constant: $\mathrm{ht}_\infty(E) \approx 6\mathrm{ht}^{\mathrm{Falt}}(E)$ with respect to $D = [\infty] \in M_{1,1}$ – ibid. Prop. 3.4.

§A Global Multiplicative Subspace. Assume furthermore the existence of a global multiplicative subspace of E_K – i.e. a one-dimensional $H \leqslant E_K[\ell]$ which at $v \in \mathbb{V}(K)$ where E has *potentially multiplicative reduction* coincides with $\mu_\ell \leqslant E_v[\ell]$ – recall that via the Tate curve one recovers $E_v \simeq \mathbb{G}_m/q_v^{\mathbb{Z}}$. Forming the isogenous curve $E_H = E_v/H$ gives the diagram in Fig. 2 (LHS), which by *Faltings' height invariance by isogeny and their log-definition*, provides a Vojta-like bound of the height inequality as in Fig. 2 (RHS) – see [GenEll] Lem. 3.5 and [Alien] §2.3.

Fig. 2. *Bounding heights Vojta-like for elliptic curves with GMSCG.*

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{(-)^\ell} & \mathbb{G}_m \\ \downarrow & & \downarrow \\ E_v \simeq \mathbb{G}_m/q_v^{\mathbb{Z}} & \xrightarrow{/H} & \mathbb{G}_m/q_v^{\ell\mathbb{Z}} \simeq E_H \end{array} \quad \Rightarrow \quad \ell.\mathrm{ht}^{\mathrm{Falt}}(E) \approx \mathrm{ht}^{\mathrm{Falt}}(E_H) \lesssim_{\mathrm{Falt}} \mathrm{ht}^{\mathrm{Falt}}(E) + \log(\ell)$$

§Two steps towards IUT Geometry. As presented in [GenEll] – then formally established by [IUTChIV] Cor. 2.2 – the existence of such a global multiplicative $\mathbb{F}_\ell$-subspace $H < E_K[\ell]$ with a canonical generator modulo $\{\pm 1\}$ ***(GMSCG)***, provides a favourable context for establishing the *Vojta Conjecture (Gen.)*

Moreover, since the Frobenius-like $(-)^\ell$ is obviously not a ring homomorphism – $(a+b)^\ell \neq a^\ell + b^\ell$ – the definition of a "global" Frobenius-like morphism will require to uncouple the $\boxtimes$ and $\boxplus$-monoid structures – see Talk 2.1 §The Log-theta-lattice of Hodge theaters – which will be done using anabelian techniques – see for example Fig. 10-11 in Talk 2.2.

At a categorical level, both the anabelian techniques and the solution to (GMSCG) require the construction of a new apparatus of *Frobenioid* and $\Theta^{\pm\mathrm{ell}}$NF-Hodge Theaters whose role is, for various $\mathrm{Spec}\, K \to \mathcal{M}_{1,1}$, to host the local-global and arithmetic-geometric symmetries or arithmetic line bundles at various places of K and in an algorithmic way – see Topic 2 and ※ in Topic 3.

※ *Recall that an elliptic curve is entirely determined by the evaluation of Weierstrass' ϑ-function on its torsions points – see [Mum83] Chap. I, a result which, on contrary to Riemann's function, does extends to p-adic fields. In the categorical constructions of IUT, elliptic curves and their arithmetic appear via their Tate q-parameters and some p-adic Θ-functions – see Tab. 3 and Talk 2.2.*

References.

[BG06] E. Bombieri and W. Gubler, *Heights in Diophantine geometry*, ser. New Mathematical Monographs. Cambridge University Press, Cambridge, 2006, vol. 4, pp. xvi+652, MR2216774.

[Del84] P. Deligne, "Preuve des conjectures de Tate et de Shafarevitch," *Séminaire Bourbaki*, vol. 26, pp. 25–41, 1983-1984, Eprint available on-line.

[Fes15] I. Fesenko, "Arithmetic deformation theory via arithmetic fundamental groups and nonarchimedean theta-functions, notes on the work of Shinichi Mochizuki," *Eur. J. Math.*, vol. 1, no. 3, pp. 405–440, 2015, MR3401899.

[GenEll] S. Mochizuki, "Arithmetic elliptic curves in general position," *Math. J. Okayama Univ.*, vol. 52, pp. 1–28, 2010, MR2589843, Eprint available on-line.

[Mum83] D. Mumford, *Tata lectures on theta. I*, ser. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1983, vol. 28, pp. xiii+235, With the assistance of C. Musili, M. Nori, E. Previato and M. Stillman, MR688651.

[Sil94] J. H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, ser. Graduate Texts in Mathematics. Springer-Verlag, New York, 1994, vol. 151, pp. xiv+525, MR1312368.

[Voj98] P. Vojta, "A more general *abc* conjecture," *Internat. Math. Res. Notices*, no. 21, pp. 1103–1116, 1998. arXiv: `math/9806171v1`, MR1663215.

The goal of this section is to develop the definitions and key properties of the IUT geometry relatively to the stack of elliptic curves. We first motivate in Talk 2.1 the categorical structures hosting the adhoc arithmetic and geometric symmetries of the context:

(i) $\mathcal{F}^{\bullet}$*-prime strips* $\mathfrak{F}^{\bullet}$ are collections of Frobenioids that mimic classes of arithmetic line bundles at various places $\mathbb{V}(K)$ of various elliptic curve $\operatorname{Spec} K \to \mathcal{M}_{1,1}$. Their construction is global-local and with respect to the ℓ-torsion points $E_K[\ell]$ of E. They can be seen as a "Galois-monoid analog of adèles and idèles";

(ii) $\Theta^{\pm ell}$*NF-Hodge theater* $\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ allow the synchronization of the arithmetic and geometric symmetries of the prime strips – i.e. with respect to $\mathrm{Gal}(K/F_{\mathrm{mod}})$ for F_{mod} field of moduli of E_K, and $\mathrm{Aut}(X_K)$ of cusps of hyperbolic curve of genus 1. They provide an answer to the (GMSCG) problem – see [IUTChI] Th. A (i) – and can be seen as some "miniature models of the geometry of the Galois monoids surrounding various arithmetic line bundles".

Hodge theaters come with Frobenius-like and étale-like layers, the latter being the anabelian context whose isomorphism classes allow the deformation of the $\boxplus/\boxtimes$-structures of the ring-scheme ones.

Talk 2.2 presents what is the core of the IUT geometry: how the properties of the Hodge theaters of Talk 2.1 and the anabelian results of Topic 3 provide an indeterminacy-compatible embedding of Frobenius-like objects into étale-like ones (aka the multiradiality or cyclotomic rigidity by Kummer theory, see [EtTh]). It comes in 3 flavors: for number and function fields, and for Θ-functions over nonarchimedean fields – aka the mono-theta environment of [IUTChII] §1. The former is a refined Local Class Field Theory, the latter is related to Mumford's construction of singular abelian varieties [Mum72].

Finally, Talk 2.3 shows how the previous constructions and the nonarchimedean logarithm, by allowing the global decoupling of the $\boxplus/\boxtimes$-monoids structures of the various $\mathcal{O}_{F_v}$s – $v \in \mathbb{V}(F_{\mathrm{mod}})$, provide **(1)** the existence of a certain log-shell region within a log-theta-lattice of Hodge theaters [IUTChIII] Th. A, that gives **(2)** a meaningful estimate of the height on Frobenius-like objects [IUTChIII] Th. B.

※ *The meaningfulness of the height estimate follows the existence of indeterminacies (Ind1), (Ind2), and (Ind3) in IUT Geometry: (Ind1) and (Ind2) come from the mono-anabelian transport* $Frob \to Etale \simeq Etale \to Frob$ *of Talk 2.2 and corresponds to some* $Aut(G_k)$ *and* $\mathcal{O}_k$*-symmetries of the prime strips; (Ind3) comes from the global symmetry of the log-shell region. The final abc-inequality à la [GenEll]-Talk 1.3 is indeed [IUTChIV] Th. A.*

※ *Notations. For* $\underline{v} \in \underline{\mathbb{V}}^{non}$, $k = F_{\underline{v}}$: $\mathcal{O}_{\bar{k}}$ *ring of integers,* $\mathcal{O}_{\bar{k}}^{\triangleright}$ *the* $\boxtimes$*-monoid of non-zero integers,* $\mathcal{O}_{\bar{k}}^{\times\boldsymbol{\mu}} = \mathcal{O}_{\bar{k}}^{\times}/\mathcal{O}_{\bar{k}}^{\boldsymbol{\mu}}$ *the* $\boxtimes$*-monoid of units (mod roots of unity), and* $\underline{q}_{\underline{v}} \in \mathcal{O}_{\bar{k}}$ *(resp.* $\underline{\underline{q}}_{\underline{v}} \in \mathcal{O}_{\bar{k}}$ *) the q-parameter (resp. a* 2ℓ*-th root of) of* E_F *at* $\underline{v}$ *– see also Tab. 3 for a global IUT overview.*

TALK 2.1 - $\Theta^{\pm ell}$NF-HODGE THEATERS: AN APPARATUS FOR GLOBAL MULTIPLICATIVE SUBSPACES.

We present the multiple structures in which evolves the IUT geometry, give some explicit examples of objects – see Tab. 1 and illustrate the main arithmetic-geometric synchronisation property of the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters.

§Prime Strips. An initial Θ-data is a 7-tuple:

$$(\bar{F}/F, X_F, \ell, \underline{C}_K, \underline{\mathbb{V}} \simeq \mathbb{V}_{\mathrm{mod}}, \mathbb{V}_{\mathrm{mod}}^{bad}, \underline{\varepsilon})$$

with F number field, X_F hyperbolic curve of type $(1,1)$, $\ell \geqslant 5$ prime, $\underline{C}_K$ hyperbolic orbicurve, $\underline{\varepsilon}$ a cusp of $\underline{C}_K$, that satisfies some technical properties – see [IUTChI] Def. 3.1 or [Alien] §3.3 (i) – the most important ones being:

(i) $\mathbb{V}_{\mathrm{mod}}^{bad}$ is a set of nonarchimedean places of the field of moduli F_{mod} of the elliptic curve E_F associated to X_F giving bad multiplicative reduction;

(ii) ℓ has ℓ-adic Galois representation $G_F \to \mathrm{Out}(E[\ell]_{\bar{F}})$ that contains $\mathrm{SL}_2(\mathbb{F}_\ell)$;

(iii) X_F is not of Shimura type.

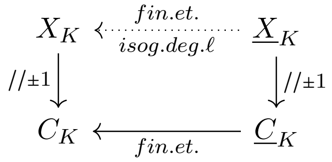
All together these assumptions implies that the top arrow of Fig. 3 arises from a rank one quotient $H < E_K[\ell] \to Q \simeq \mathbb{Z}/\ell\mathbb{Z}$ such that:

(i) H coincides with the canonical subspace at $v \in \underline{\mathbb{V}} \simeq \mathbb{V}_{mod}$;

(ii) the cusp ε of $\underline{C}_K$ coincides with the canonical generator at $v \in \underline{\mathbb{V}}$.

In this sense $(\underline{C}_K, \varepsilon)$ simulate the global multiplicative subspace and a canonical generator modulo $\{\pm 1\}$ – see motivation (GMSCG) of Talk 1.3. Note also that (iii) implies that the orbicurve $C_F = X_F//\{\pm 1\}$ is a final object in the category of local étale systems $\overline{\mathrm{Loc}}_F(C)$ (i.e. C_F is F-coric) – see Talk 3.3.

Fig. 3. *Cusps & Can. Generators*



The notion of line bundles (or monoid or divisor) on hyperbolic curves in relation with the Θ-functions of Talk 1.3 is abstractly encoded in the categorical notion of Frobenioid (~ 2005.) – we refer to [IUTChI] Ex. 3.2 and to Talk 2.2 *§Theta-function as Symmetric Line Bundle* for an illustration of this fact. In the case of IUT, one can indeed restrict to *Frobenioids of nonarchimedean type* (i.e. $\underline{v} \in \underline{\mathbb{V}}^{non}$) and their global realified versions as given in Tab. 1 – where $\Pi_X^{(\mathrm{temp})}$ acts as an abstract topological group via $\Pi_X \to G_{\underline{v}}$, see also [Alien] §3.5.

A $\mathcal{F}^{\bullet}$*-prime strip* is then a collection of data-Frobenioids and isomorphims indexed by $\mathbb{V}$ that are equivalent to certain models allowing to recover the given initial Θ-data. For example and more formally

$$\mathfrak{F}^{\vdash\times} = \{\mathcal{F}^{\times}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} \text{ and } \mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}} = \{\mathcal{C}^{\Vdash}, \mathrm{Prime}(\mathcal{C}^{\Vdash}) \simeq \mathbb{V}, \mathfrak{F}, \{\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}\}$$

where $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$ is a *global realified* version of $\mathcal{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}$ with $\mathcal{C}^{\Vdash}$ encoding some "arithmetic divisors" Φ of F_{mod} of the form

$$\Phi = \bigoplus_{v\in\mathbb{V}^{non}} \mathrm{ord}(\mathcal{O}_v^{\triangleright}) \otimes \mathbb{R}_{\geqslant 0} \bigoplus_{v\in\mathbb{V}^{arc}} \mathrm{ord}(\mathcal{O}_v^{\triangleright})$$

Tab. 1. *Types of $\mathcal{F}^{\square}$-prime strips and local data at $\underline{v} \in \mathbb{V}^{bad}$.*

$\mathcal{F}^{\vdash}$	$G_{\underline{v}} \curvearrowright \mathcal{O}_{\bar{F}_{\underline{v}}}^{\times} \times q_{\underline{v}}^{\mathbb{N}}$
$\mathcal{F}^{\vdash\times}$	$G_{\underline{v}} \curvearrowright \mathcal{O}_{\bar{F}_{\underline{v}}}^{\times}$
$\mathcal{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}$	$G_{\underline{v}} \curvearrowright \mathcal{O}_{\bar{F}_{\underline{v}}}^{\times\boldsymbol{\mu}} \times q_{\underline{v}}^{\mathbb{N}}$
$\mathcal{F}$	$G_{\underline{v}} \curvearrowright \Pi_{\underline{v}}^{\mathrm{temp}}$
$\mathcal{D}^{\vdash}$	$G_{\underline{v}}$
$\mathcal{D}$	$\Pi_{\underline{v}}^{\mathrm{temp}}$

Notations: $G_{\underline{v}} = \mathrm{Gal}(\bar{k}/k)$, $k = K_{\underline{v}}$, $\mathcal{F}^{\Vdash\square}$ realified version of $\mathcal{F}^{\vdash\square}$.

and the $\{\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ are "global-to-local" isomorphisms $\rho_{\underline{v}}\colon \Phi_v \xrightarrow{\sim} \mathbb{R}_{\geq 0}$ – see [IUTChI] Ex. 3.8.

In order to minimize the weight of this formalism, one can think of a prime strip as a collection of $G_{\underline{v}}$- and $\Pi_{\underline{v}}$-topological monoids $\mathcal{O}_{\underline{v}}^{\bullet}$ attached to the initial Θ-data at $v \in \mathbb{V}^{bad}$ – see Tab. 1 for bad places and Tab. 2 for others, also [IUTChI] Fig. I1.2 for a list of $\mathfrak{F}^{\square}$-prime strips with references.

Note that, similarly to divisors, $\mathcal{F}^{\bullet}$-prime strip are defined relatively to some base category $\mathcal{D} = \{\Pi_v\}_{v\in\mathbb{V}}$ or $\mathcal{D}^{\vdash} = \{G_v\}_{v\in\mathbb{V}}$ or $\mathfrak{D}$- and $\mathfrak{D}^{\vdash}$-prime strips.

※ *The key data in the Θ-data definition is indeed the cusp $\underline{\varepsilon}$; its existence here follows from the* $\mathrm{SL}_2(\mathbb{F}_\ell)$ *property (ii) – see [IUTChIV] p. 46.*

§The $\Theta^{\pm ell}$***NF-Hodge Theaters and Symmetries.*** A $\Theta^{\pm\mathrm{ell}}$NF-Hodge Theater $\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ is *essentially* a system of Frobenioids obtained by gluing a $\Theta^{\pm\mathrm{ell}}$-Hodge Theater $\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}}$ and a ΘNF-Hodge Theater $\mathcal{HT}^{\Theta\mathrm{NF}}$ together

$$\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} = \{\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}}, \mathcal{HT}^{\mathrm{NF}}, \text{glue isom.}\}$$

– see [IUTChI] Def. 6.13 (i). Roughly and formally, that is:

$$\mathcal{HT}^{\Theta\mathrm{NF}} = \begin{cases} \bullet\text{A } \Theta\text{-Hodge theater } \mathcal{HT}^{\Theta} = (\{\mathcal{F}_v\}_{\underline{\mathbb{V}}}, \mathfrak{F}^{\Vdash}_{\mathrm{mod}}) \\ \bullet\text{A } \mathcal{F}\text{-prime stripe } \mathfrak{F}_{>} = \{\mathcal{F}_{>,\underline{v}}\}_{\underline{\mathbb{V}}} \\ \bullet\text{A capsule } \mathfrak{F}_J = \{\mathfrak{F}_j\}_J \text{ of } \mathcal{F}\text{-prime strips indexed by } J \simeq \mathbb{F}_\ell^{*} \end{cases} ; \quad \mathcal{HT}^{\Theta\pm\mathrm{ell}} = \begin{cases} \bullet\text{A } \mathcal{F}-\text{prime stripe } \mathfrak{F}_{>} = \{\mathcal{F}_{>,\underline{v}}\}_{\underline{\mathbb{V}}} \\ \bullet\text{A capsule } \mathfrak{F}_T = \{\mathfrak{F}_i\}_T \text{ of } \mathcal{F}\text{-prime strips indexed by } T \simeq \mathbb{F}_\ell \end{cases}$$

The $\mathcal{HT}^{\Theta\mathrm{NF}}$ is of *arithmetic nature* and related to number field, while $\mathcal{HT}^{\Theta\pm\mathrm{ell}}$ is of *geometric nature* and related to elliptic curves. The former includes two global portions, a realified one and a container for the cups of X_K; the global portion of the latter is a container for the places of K/F_{mod} – see Fig. 4. Both are indeed based on some Hodge theater versions of respectively the $\mathfrak{D}$ and $\mathfrak{D}^{\vdash}$-prime strips – see also [IUTChI] Fig. 6.6 for a list of Frobenioids.

In terms of GMSCG, the key property of $\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$ is to host the multiple avatars of the global multiplicative subspace $H \simeq \mathbb{F}_\ell$ modulo $\{\pm 1\}$ by *allowing the global synchronization of two kinds of symmetries and of the local $\{\pm 1\}$-indeterminacies of the primes stripes*:

(i) *Geometric symmetries:* the set of cusps of $\underline{X}_K$ identified to $Q \simeq \mathbb{F}_\ell$ carries some $\text{Aut}(X_K) \approx \mathbb{F}_\ell^{\rtimes\pm}$-symmetries;
(ii) *Arithmetic symmetries:* the orbicurve $\underline{C}_K$ carries the symmetries of the Galois group $\mathbb{F}_\ell^{\circledast} \approx \text{Gal}(K/F_{\text{mod}}) \hookleftarrow \text{Aut}(\underline{C}_K)$;
(iii) *Global symmetries:* In terms of the Galois representation of the ℓ-torsion points, $\mathbb{F}_\ell^{\circledast} \simeq \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) / \left(\begin{smallmatrix} * & * \\ 0 & \pm 1 \end{smallmatrix}\right)$.

Here one denotes $\mathbb{F}_\ell^{\rtimes} = \mathbb{F}_\ell \rtimes \{\pm 1\}$ and $\mathbb{F}_\ell^{\circledast} = \mathbb{F}_\ell^{\times}/\{\pm 1\}$, with the actions $\mathbb{F}_\ell^{\rtimes} \curvearrowright \mathbb{F}_\ell$ and $\mathbb{F}_\ell^{\circledast} \curvearrowright \mathbb{F}_\ell^{\circledast}$, and $\ell^{\circledast} = (l-1)/2$.

We refer to Fig. 4 for the gluing of the $\mathcal{HT}^{\Theta^{\pm\text{ell}}}$ and $\mathcal{HT}^{\text{NF}}$ theaters into a $\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$, see also the discussion in [Alien] §3.3 (iv) and (v), to Fig. 5 in terms of labels $\mathbb{F}_\ell^{\times}$ and $\mathbb{F}_\ell^{\circledast}$, and to Fig. 6 in terms of $\mathcal{D}$-base of Hodge Theaters, and also [IUTChI] Th. A (i).

Fig. 4. *Geometric and Arithmetic Symmetries of a theatre $\mathcal{HT}^{\Theta^{\pm\text{ell}}}$: Global synchronization of the $\{\pm 1\}$-indeterminacies of a $\mathcal{F}$-prime strip via the labels – many flavours. Notations: $\succ$ and $>$ denote different classes of $\mathfrak{D}$-strips.*

Frobenioids labels	$\succ := [-l^{\circledast} < \cdots < -1 < 0 < 1 < \cdots < l^{\circledast}]$	$\xleftarrow[\mathbb{F}_\ell^{\times} \to \mathbb{F}_\ell^{\circledast},\ \pm t \mapsto \lvert t \rvert]{\text{Glue}}$	$[1 < 2 < \cdots < l^{\circledast} - 1 < l^{\circledast}] := >$
Local: $SL_2(\mathbb{F}_\ell)$	$\begin{pmatrix} * & * \\ 0 & \pm 1 \end{pmatrix} \mathbb{V}$		$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mathbb{V}$
Global:	$\mathbb{F}_\ell^{\rtimes\pm} \approx \text{Aut}_K(\underline{X}_K) \hookrightarrow \Pi_{\underline{X}_K} \curvearrowright \{\text{Cusps of } \underline{X}_K\}$		$Aut(\underline{C}_K) \hookrightarrow \text{Gal}(K/F_{\text{mod}}) \approx \mathbb{F}_\ell^{\circledast}$
Symmetries	Geometric & Additive		Arithmetic & Multiplicative

– see also Fig. 5 and Fig. 6 for definitions in terms of labels or of $\Theta^{\pm\text{ell}}$NF-Hodge theaters.

In conclusion, and with the words of [Alien] §3.3 (iv), a $\Theta^{\pm\text{ell}}$NF-Hodge Theater can be thought as a system of Frobenoids-data – such as *(i) topological groups $\pi_1^{(\text{temp})}(X_F)$, (ii) rational function monoids $\mathcal{O}_{K^*_{(v)}}$ for K extension of F_{mod} (completed at $v \in \mathbb{V}$)*, and *(iii) monoids of effective arithmetic divisors as in Talk 1.3.* – whose structures allows the global synchronization of local symmetries.

Tab. 2. *Typical Frob. data of a $\Theta^{\pm\text{ell}}$NF-Hodge Theater.*

$\mathbb{V}^{bad,non}$	$\Pi_X \curvearrowright \mathcal{O}_k^{\triangleright}$
$\mathbb{V}^{bad}$	$\Pi_X^{\text{temp}} \curvearrowright \mathcal{O}_k^{\triangleright}$
$\mathbb{V}^{arc}$	Aut-holom.

※ *Two anabelian remarks – see Talk 3.2: (1) Via the identification of the prime arithmetic ramification to the divisorial geometric one, anabelian reconstructions theorems provides the reconstruction of prime-valuation properties; (2) Via $\Pi^{\text{temp}}_{\underline{X}_{\underline{v}}} \to G_{\underline{v}}$, the geometric $\mathbb{F}_\ell^{\rtimes\pm}$-symmetries on the labelled $\{G_{\underline{v}}\}_{t \in \mathbb{F}_\ell}$s correspond to the $\Pi^{\text{temp}}_{\underline{X}_{\underline{v}}}$-conjugacy of cuspidal inertia groups. The construction above thus induces an automatic "conjugate synchronization" of the label with respect to the places – see [Alien] §3.6 (ii), and later Talk 2.3.*

Fig. 5. *Combinatorial gluing of $\Theta^{\pm ell}$NF-Hodge theaters along labels – [Alien] Fig. 3.8.*

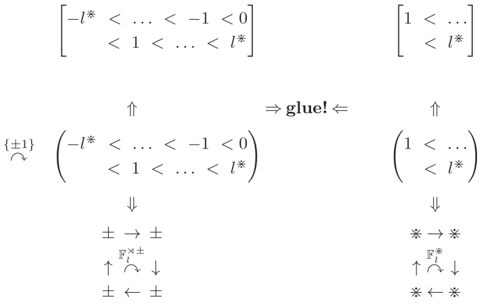
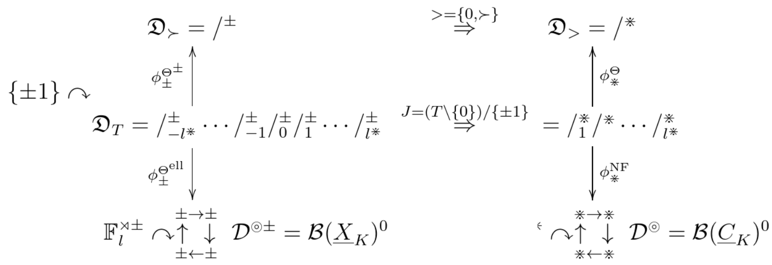


Fig. 6. *Gluing of the $\mathcal{D}$-bases of $\Theta^{\pm ell}$NF-Hodge Theaters; $/_i^{\square}$ denotes the prime stripe of label $i \in T$ of additive or multiplicative type for $\square = \pm$ or $\square = \circledast$ – [Yam17] p. 143.*



§The Log-theta-lattice of Hodge Theaters. The Θ-link is a morphism *à la* Frobenius – i.e. that mimics the $(-)^{\ell}$-morphism of Talk 1.3 – at the level of $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters, while log-links "juggle" the multiplicative and additive structures. By allowing some mild indeterminacies, they all together allow the decoupling of the $\boxplus/\boxtimes$-monoids structures. We present the essential construction and properties of log-theta-lattice of Hodge theaters that relies on the Θ-constructions of Talk 2.2 and brings the required indeterminacies for an abc-estimate in Talk 2.3.

The Θ-*link.* In its most elementary form, the Θ-link is an application between two isomorphic versions $^{\dagger}(-)$ and $^{\ddagger}(-)$ of a same Θ-Hodge theater

$$\begin{matrix} {}^{\dagger}\mathcal{HT}^{\Theta} = (\{{}^{\dagger}\mathcal{F}_v\}_{\underline{\mathbb{V}}}, {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}) \xrightarrow{\Theta} {}^{\ddagger}\mathcal{HT}^{\Theta} = (\{{}^{\ddagger}\mathcal{F}_v\}_{\underline{\mathbb{V}}}, {}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}) \\ \text{defined via } \mathcal{F}^{\Vdash}\text{-prime strips } {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}} \xrightarrow{\sim}{}^{poly}\, {}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}} \end{matrix}; \text{ with } v \in \mathbb{V}^{bad} \text{ in: } \begin{matrix} {}^{\square}\mathfrak{F}^{\Vdash}_{\mathrm{mod}} : 2\ell\text{th-root of } q_v \in \mathcal{O}_v^{\rhd} \\ {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}} : \ell\text{th-root of } \ddot{\Theta}_v \in \mathcal{O}_v^{\rhd} \end{matrix}$$

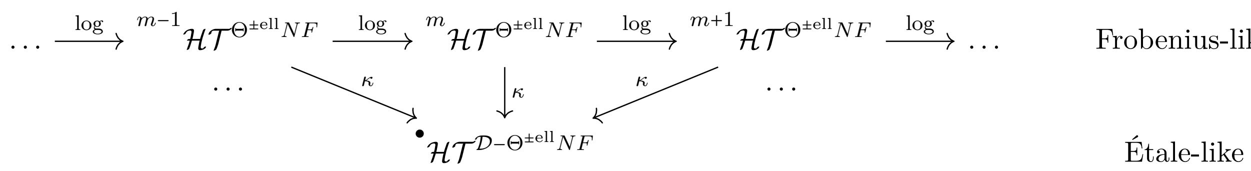
where Θ_v is the nonarchimedean Θ-function of Talk 2.2 (in its Frobenioid version). The Θ-link can thus be seen as $\{\Theta_v^{\mathbb{N}} \mapsto q_v^{\mathbb{N}}\}_{\mathbb{V}^{bad}}$ and as (i) *dilating the value groups* $\mathcal{O}_v^{\rhd}$, $v \in \mathbb{V}^{bad}$, while (ii) *preserving the unit group* $\mathcal{O}^{\times}_{{}^{\dagger}C_v} \simeq \mathcal{O}^{\times}_{{}^{\ddagger}C_v}$, $v \in \mathbb{V}^{good,\ bad}$ – see [IUTChI] Th. A (ii) or ibid. Cor. 3.7 (i) and (iii). An iteration of the Θ-link between Hodge theaters then provides the following *Frobenius picture of Hodge theaters* – see ibid. Cor. 3.7 (ii) and 3.8:

$$\ldots \xrightarrow{\Theta} {}^{n-1}\mathcal{HT}^{\Theta} \xrightarrow{\Theta} {}^{n}\mathcal{HT}^{\Theta} \xrightarrow{\Theta} {}^{n+1}\mathcal{HT}^{\Theta} \xrightarrow{\Theta} \ldots$$

Considering the $\mathcal{D}$-bases $\mathcal{D}$-Θ-Hodge theaters, the unique isomorphic class ${}^{\bullet}\mathcal{D}^{\vdash} = {}^{n}\mathcal{D}^{\vdash} \simeq {}^{n+1}\mathcal{D}^{\vdash}$ provides a similar 2-dimensional étale picture between ${}^{n}\mathcal{D}_v$s and ${}^{\bullet}\mathcal{D}_v^{\vdash}$s, see Cor. 3.9.

The log-*link.* In another direction, the p-adic $\log\colon \mathcal{O}_v^{\times} \to \bar{k}_v$ defines a Π_v-equivariant isomorphism of Ind-topological module $\mathcal{O}_{\bar{k}}^{\times\mu} \otimes \mathbb{Q} \simeq \bar{k}_v$ – i.e. a *topological compatibility*, see Talk 2.2 – that first passes to the $\mathcal{F}$-prime strips $\mathfrak{log}\colon {}^{\dagger}\mathfrak{F} \to {}^{\ddagger}\mathfrak{F}$ then to the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters $\mathfrak{log}\colon {}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}NF} \to {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}NF}$ with isomorphisms between $\mathcal{D}$-prime strips, see [IUTChIII] Prop. 1.2 (i), and Prop. 1.3 (i)-(ii) coricity. One obtains a non-commutative *vertical Frobenius picture* – see ibid. (iv):

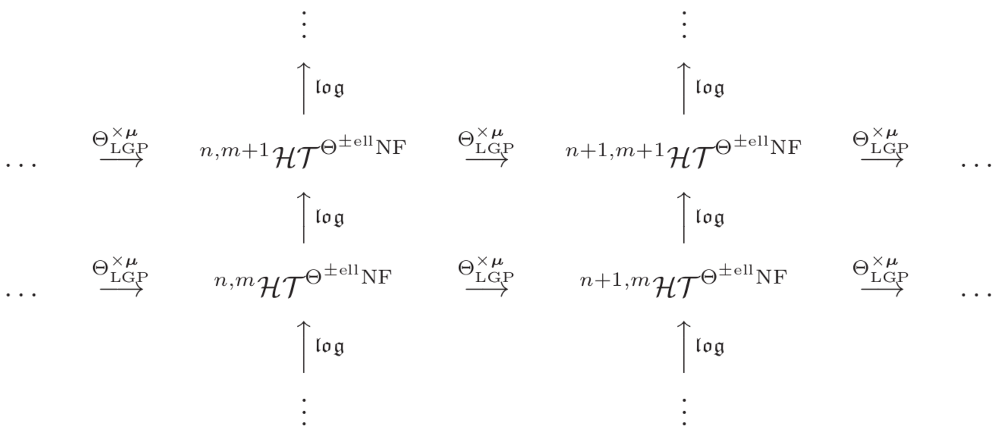
Fig. 7. *Non-commutative Frobenius-étale pictures of $\Theta^{\pm ell}$NF-Hodge theaters.*



where the κs are induced by the Kummer embeddings of Talks 2.2 and 3.3.

This definition of the Θ-link first extends in a $\Theta^{\times\boldsymbol{\mu}}_{Gau}$-link with respect to $\Theta^{\pm\mathrm{ell}}$NF-Hodge Theaters and $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime strips [IUTChII] Cor. 4.10 (iii), then in a $\Theta^{\times\boldsymbol{\mu}}_{LGP}$-link [IUTChIII] Def. 3.8 (ii) that is relative to the $\mathfrak{log}$-link above – see ibid. (iii). The Frobenius-Étale picture above then takes place vertically in a 2-dimensional lattice, where **(1)** each column stands on top of *a common étale container* ${}^{\bullet}\mathcal{HT}^{\mathcal{D}-\Theta^{\pm\mathrm{ell}}NF}$ that **(2)** is not shared via the Θ-links – see Fig. 8 below. The ${}^{\bullet}\mathcal{HT}^{\mathcal{D}-\Theta^{\pm\mathrm{ell}}NF}$ is said to be $\mathcal{D}$-holomorphic in reference to (1), and will later be transformed in a $\mathcal{D}^{\vdash}$-mono-analytic objects – see Tab. 14 – *as consequences of the mono-anabelian constructions of Talk 3.3*.

Fig. 8. *The LGP-Gaussian log-theta lattice as in [IUTChIII] Def. 1.4*



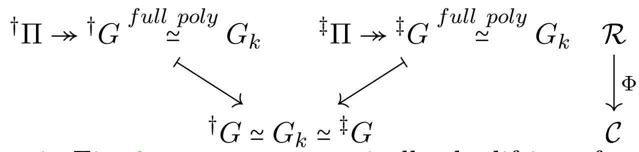
In this picture, the étale-like ${}^{\bullet}\mathcal{HT}^{\mathcal{D}-\Theta^{\pm\mathrm{ell}}NF}$ is said to be *vertically coric*. The comparison of two sides of the Θ-link after $\mathfrak{log}$-juggling requires some *multiradial properties* – see Talk 2.2 – as well as the introduction of mild indeterminacies (Ind1), (Ind2) and (Ind3) for a log-Θ-wandering – see Talk 2.3.

※ *In the Frobenius-Étale picture Fig. 7: the first row is* Frobenius-like objects *– since related to the q_vs, while the second row is* Étale-like ones *– since related to anabelian Π_vs; Reversing the Kummer arrows provides the embedding of Galois objects into some global analytic containers, a picture similar to Ihara's foundation of Grothendieck-Teichmüller theory. LGP stands for "Log-arithmetic Gaussian Procession" – see Talk 2.3.*

Talk 2.2 - Cyclotomic Rigidity and Multiradiality. The dissociation of ⊠/⊞-monoids structures at the level of $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters relies at the level of objects on a functorial Kummer embeddings of rigid Frobenius-like objects into étale-like ones. This embedding is led via a cyclotomic isomorphism whose rigidity or symmetries determine a certain degree of freedom that allows the compatible continuous anabelian (re/de)constructions of the étale container. At a categorical level it determines a *uni/multiradiality* property of certain radial environments, which joined to a *topological compatibility* one allows the "log-theta-wandering" of Talk 2.3.

Formally a radial environment $\{\mathcal{R},\mathcal{C},\mathcal{R}\overset{\Phi}{\to}\mathcal{C}\}$ – composed of a "fine" coric category $\mathcal{C}$, a radial category $\mathcal{R}$, and a radial functor Φ – is said to be *multiradial* if Φ is full – see [IUTChII] Ex. 1.7-1.8 for examples and definitions, [Alien] §3.1 for discussion and §3.2.2 §3.2.1, and below for examples. Note that considering $\mathrm{Aut}(G)$-orbits instead of poly-isomorphisms in $\mathcal{R}$ as in Fig. 9 ensures automatically the lifting of isomorphisms from $\mathcal{C}$ to $\mathcal{R}$.

Fig. 9. *Multiradiality - Ex. 1.8 (i) [IUTChII] – ${}^{\dagger}(-)$ and ${}^{\ddagger}(-)$ denotes different version of isomorphic groups.*



Following Grothendieck's idea of crystals, a multiradial environment can be thought as a "fibration endowed with a connection whose monodromy allows parallel transport of structures between the fibers".

We present the *multiradial* and *topological compatibility* properties for various portions of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime strips – see Tab. 3 for a global overview:

(i) For local places in $\mathbb{V}^{good,non}$ in terms of the local unit group $\mathcal{O}_{\bar{k}}^{\triangleright}$ or its quotient $\mathcal{O}_{\bar{k}}^{\times\mu}$, via *Brauer group techniques* – see Fig. 10 and [IUTChII] §1;

(ii) For the global portion, or F_{mod}, via κ-coric functions and *anabelian geometry* – see Talk 3.3 and [IUTChI] Ex. 5.1 (v);

(iii) For local places in $\mathbb{V}^{bad}$ in terms of the local value group generated by $\{q^{j^2}\}_j \leqslant \mathcal{O}_{\bar{k}}^{\triangleright}$, via mono-theta environments and the theory of *nonarchimedean* Θ*-functions* of [EtTh] §1-2.

A construction is said to be topological compatible if it is with respect to the G_k-ind one of the monoids – e.g. for $\mathcal{O}_{\bar{k}}^{\triangleright}$ with respect to the ind-limit of $\{\mathcal{O}_{\bar{k}^H}^{\triangleright},\ H < G_{\bar{k}} \text{ cofinite}\}$.

The case (iii) relies on three rigidity properties of a specifically defined mono-theta environment $\mathbb{M}^{\Theta}$ related to *nonarchimedean* Θ*-functions* – see [Alien] §3.4 (iii)-(iv) – which for $v \in \mathbb{V}^{non,\ bad}$ are defined by:

$$\ddot{\Theta}_{\underline{v}}(\ddot{U}_{\underline{v}}) := q_{\underline{v}}^{-1/8} \sum_{n\in\mathbb{Z}} (-1)^n \cdot q_{\underline{v}}^{\frac{1}{2}(n+\frac{1}{2})^2} \cdot \ddot{U}_{\underline{v}}^{2n+1} \text{ with sym. } \begin{array}{l} \ddot{\Theta}(\ddot{U}_{\underline{v}}) = -\ddot{\Theta}(\ddot{U}_{\underline{v}}^{-1});\ \ddot{\Theta}(-\ddot{U}_{\underline{v}}) = -\ddot{\Theta}(\ddot{U}_{\underline{v}}) \\ \ddot{\Theta}(q^{j/2}\ddot{U}) = (-1)^j q^{-j^2/2}\ddot{U}^{-2j}.\ddot{\Theta}(\ddot{U}) \end{array} \quad (1)$$

– see also [EtTh] Prop. 1.4. We review the construction of this framework whose three rigidity properties lead to the multiradiality and topological compatibility of $\mathbb{M}^{\Theta}$.

※ *The whole process of cyclotomic rigidity can also be understood in terms of mono-anabelian transport as illustrated in Fig. 10 and 11: note that the diagrams commute up-to some $\widehat{\mathbb{Z}}^{\times}$- and $\{\pm1\}$-indeterminacy. – see mono-theta environments of Gaussian Monoids in [IUTChIII] §2-§4, and Talk 2.3. Technical note: For (ii) the Galois evaluation of [Alien] 3.6 recovers the finite extensions of F_{mod}.*

§Rigidity via Brauer and κ-coric functions. To $M \curvearrowleft G$ a Frobenius-like topological multiplicative monoid containing some torsion $\boldsymbol{\mu}(M) < M$ – e.g. $\mathcal{O}_{\bar{k}}^{\triangleright} \curvearrowleft G_k$ or $\bar{K}^{\times} \curvearrowleft G_K$ – one attaches functorially a Kummer morphism:

$$M \xrightarrow{\kappa} H^1(G, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(M)) \text{ where } \begin{cases} H^1(G, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(M)) := \varinjlim H^1(G, \boldsymbol{\mu}_{\mathbb{Z}/N\mathbb{Z}}(M)) & \text{is étale-like} \\ \boldsymbol{\mu}_{\mathbb{Z}/N\mathbb{Z}}(M) = Hom(\mathbb{Z}/N\mathbb{Z}, M_{tors}) & \text{is Frobenius-like.} \end{cases}$$

The $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(-) \simeq \widehat{\mathbb{Z}}(1)$ are *cyclotomes* that come with *cyclotomic rigidity isomorphism* $\rho: \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(M) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G)$ – see Talk 3.3 *§Absolute mono-anabelian reconstructions (1)* – which are canonically defined up to $\mathrm{Aut}(\widehat{\mathbb{Z}}) \simeq \widehat{\mathbb{Z}}^{\times}$. We discuss, for various radial environments, how symmetries/indeterminacies of the cyclotomic rigidity isomorphisms in terms of topological compatibility and multiradiality – see discussion of [Alien] §2.6.1 (i)-(iii).

For case (i), we refer to the functorial construction of [Alien] Ex. 2.12.1-2 in terms of Brauer groups, which, it is shown, either admits a $\widehat{\mathbb{Z}}^{\times}$-symmetry or is topological compatible. The radial environment is given by $(\mathcal{R}, \mathcal{C}, \Phi)$ with some abstract $\mathcal{R} = \mathcal{O}_{\bar{k}}^{\triangleright} \curvearrowleft G_k$, $\mathcal{C} = \mathcal{O}_{\bar{k}}^{\times\boldsymbol{\mu}} \curvearrowleft G_k$, and Φ the quotient $\mathcal{O}_{\bar{k}}^{\triangleright} \to \mathcal{O}_{\bar{k}}^{\times\boldsymbol{\mu}}$ whose potential $\widehat{\mathbb{Z}}^{\times}$-symmetries are incompatible with ρ – see [Alien] §3.4 (i). The case (i) is thus *topological compatible but not multiradial*.

For case (ii), we refer to the κ-coric rational functions of [Alien] §2.13.1. For K_X function field of a hyperbolic curve X over k – k number or sub-p-adic field, the Kummer morphism $K_X \to H^1(G_{K_X}, \boldsymbol{\mu}(\bar{K}_X))$ is associated to $\rho: \boldsymbol{\mu}(\bar{K}_X) \xrightarrow{\sim} \boldsymbol{\mu}(G_{K_X})$ is a $\widehat{\mathbb{Z}}^{\times}$-torsor under the inertia groups I_x of closed points x of $X(k)$. The radial data are given as in Fig. 9 and the evaluation provides a $\{\pm1\}$-indeterminacy for ρ that is *multiradial but not topological compatible* – see discussion [Alien] §3.4 (ii), and [IUTChI] Ex. 5.1 (v) for Frobenioids .

Fig. 10. *Cyclotomic rigidity and Mono-anabelian trans. in $\mathbb{V}^{non}$ via Brauer.* $* = Aut(G_k \curvearrowright \mathcal{O}^\times_{\bar{k}})$, $\widetilde{H}^1 = Im[\kappa]$.

$$\begin{array}{cccc} \text{Ét.} & \widetilde{H}^1(G_k, \boldsymbol{\mu}^{\widehat{\mathbb{Z}}}_{\bar{k}}(G_k)) & \xrightarrow{\sim *} & \widetilde{H}^1(G_k, \boldsymbol{\mu}^{\widehat{\mathbb{Z}}}_{\bar{k}}(G_k)) \\ & \kappa\uparrow & \circlearrowleft^{\widehat{\mathbb{Z}}^\times} & \downarrow\kappa^{-1} \\ \text{Frob.} & \mathcal{O}^\times_{\bar{k}} & & \mathcal{O}^\times_{\bar{k}} \end{array}$$

Fig. 11. *Cyclotomic rigidity and Mono-anabelian transport in $F^\times_{mod}$ via κ-coricity.* $* = Aut(G_k \curvearrowright \bar{K}^\times_X)$.

$$\begin{array}{cccc} \bar{K}^\times_X(\Pi_X) & \xrightarrow{\sim *} & \bar{K}^\times_X(\Pi_X) & \text{Ét.} \\ \kappa\uparrow & \circlearrowleft^{\{\pm 1\}} & \downarrow\kappa^{-1} & \\ \bar{K}^\times_X & & \bar{K}^\times_X & \text{Frob.} \end{array}$$

※ *The multiradiality property follows in both cases the possibility of the monoid to be sent in $\mathbb{Q}_{\geq 0} \cap \widehat{\mathbb{Z}}^\times = \{1\} \subset \mathbb{Q} \otimes \widehat{\mathbb{Z}}$ – e.g. via $\mathcal{O}^\triangleright_{\bar{k}} \twoheadrightarrow \mathbb{N} \simeq \mathcal{O}^\triangleright_{\bar{k}} / \mathcal{O}^\times_{\bar{k}}$ in case of no topology compatibilty, or $K^\times_X \twoheadrightarrow \mathbb{Z}$ via the valuations at closed points of X^{cp}.*

Case (iii) requires first the construction of some mono-theta environments that we describe below.

§Theta-function as Symmetric Line Bundle. Let $K/\mathbb{Q}_p$ be a finite extension, X/K a curve of type $(1,1)$ with multiplicative stable reduction, and $\mathcal{X}^{log}$ the corresponding stable log-curve over $\mathcal{O}_K$ (with residue field k). Let us consider an $(\mu_2, \underline{\mathbb{Z}})$-étale tempered cover $\ddot{\mathcal{Y}}^{log} \xrightarrow{\mu_2} \mathcal{Y}^{log} \xrightarrow{\underline{\mathbb{Z}}} \mathcal{X}^{log}$ with generic fibres denoted $\ddot{Y}$ (resp. Y), and such that $\mathcal{L} = \mathcal{O}_{\mathcal{X}}$ induces some line bundles $\mathcal{L}_{\ddot{Y}}$ and $\mathcal{L}_Y$ by pull-back – see Fig. 12 & Fig. 19.
In terms of $\mathbb{G}_m$-torsor and $\ddot{U}$-coordinates, the $\ddot{\Theta}$-function of Eq. (1) is recovered as *quotient of an algebraic section by a theta one* – see [Alien] §3.4 (iv) & (iii), [EtTh] Prop. 1.4, and Fig. 12. More precisely, the values of $\ddot{\Theta}$ are recovered via certain Kummer cohomology classes eg $\mathcal{O}^\times_{\bar{K}}.\ddot{\eta}^\Theta \in H^1(\underline{\Pi}^{\text{temp}}_{\ddot{Y}}, \underline{\Delta}_\Theta)$ see ibid. Prop. 1.3 & 1.4.

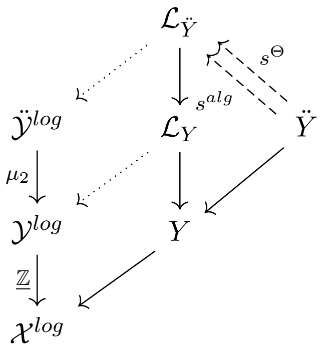


Fig. 12. *The context of mono-theta environment*

Obstructions to canonicity: (i) the section s^{alg} is defined up-to-a $K^\times$-multiple (see line bundle), (ii) the choice of base point implies a $\underline{\mathbb{Z}}$-symmetry (a priori not compatible with the (s^{alg}, s^Θ)-construction.)

§The Mono-theta Environment. Applying the tempered fundamental group and taking the N-quotient in the context above provide the *geometric model for the definition of a mono-theta environment* $\mathbb{M}^\Theta$ – see [EtTh] Def. 2.13, that is:

(i) A topological group $\pi^{\text{temp}}_{\mathcal{L}^\times_Y}[N]$;

(ii) A subgroup of symmetries $\langle \underline{\mathbb{Z}}, K^\times \rangle \leqslant Out(\pi^{\text{temp}}_{\mathcal{L}^\times_Y}[N])$;

(iii) A theta-section $s^\Theta : \pi_{\ddot{Y}} \to \pi^{\text{temp}}_{\mathcal{L}^\times_Y}[N]$ that is given up-to $\boldsymbol{\mu}_N$-inner isomorphism.

One prove such a triple of data to be **(i)** $K^\times$-invariant or *constant multiplicatively rigid*, **(ii)** compatible with the $\underline{\mathbb{Z}}$-symmetries or *discrete rigid*, and **(iii)** *cyclotomic rigid* (see below) – see [EtTh] Cor. 2.19. Note that $\underline{\mathbb{Z}}$-symmetry of $\mathbb{M}^\Theta$ corresponds to a multiplication by $(-1)^j q^{j^2/2} \ddot{U}^{-2j}$ on $\ddot{\Theta}$.

§Cyclotomic Rigidity and Multiradiality. The tempered fundamental group of $\mathbb{G}_m$-torsors provides an arithmetic-geometric fundamental sequence that is exact in our case of Serre "goodness" [Moc03b] §4.1, and one can consider the Mumford theta-group π^Θ as in Diag. 2 (LHS) – Note that π^Θ_Y corresponds to the $\widehat{\mathbb{Z}}(1) \oplus \mathbb{Z} \to \underline{\mathbb{Z}}$-quotient in $1 \to \Delta_\Theta \simeq \widehat{\mathbb{Z}}(1) \to \pi^\Theta_X \to \pi^{ab}_X \simeq \widehat{\mathbb{Z}}(1) \oplus \mathbb{Z} \to 1$.

$$\begin{array}{ccccccccc} 1 & \to & \boldsymbol{\mu}_N & \to & \pi_{\mathcal{L}^\times_Y}[N] & \to & \pi_Y & \to & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \to & \boldsymbol{\mu}_N & \to & \pi^\Theta_{\mathcal{L}^\times_Y}[N] & \to & \pi^\Theta_X & \to & 1 \end{array} \qquad \begin{array}{ccccccccc} 1 & \to & \mu_N & \to & \pi^\Theta_{\mathcal{L}^\times_Y}[N]_{|\Delta_\Theta} & \to & \Delta_\Theta & \to & 1 \\ & & & \searrow & & \swarrow & & & \\ & & & & \boldsymbol{\mu}_N \oplus \Delta_\Theta & & & & \end{array} \tag{2}$$

It follows Diag. 2 (RHS) that one recovers *canonically* the Frobenius-like and étale-like cyclotomes Δ_Θ and $\boldsymbol{\mu}_N$, as well as the cyclotomic rigidity isomorphism $\Delta_\Theta \xrightarrow{\sim} \boldsymbol{\mu}_N$ as $(1,1) \in \boldsymbol{\mu}_N \oplus \Delta_\Theta$.

The coric datas are roughly of the form $\Pi \curvearrowright \mathbb{M}^{\Theta} \xrightarrow{\Phi} G \curvearrowright \mathcal{O}^{\times\mu}$ which one shows are *multiradial* [IUTChII] Cor. 1.10 and 1.12 and *topological compatible* by their very construction. The $\mathbb{M}^{\Theta}$-cyclotomic rigidity isomorphism is also stable under the $\mathbb{F}_{\ell}^{\rtimes\pm}$-symmetry of the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters as $\mathbb{F}_{\ell}^{\rtimes\pm} \simeq \Delta_C(\mathbb{M}^{\Theta})/\Delta_X(\mathbb{M}^{\Theta})$ – where $\Delta_{\bullet}$ is the usual kernel of $\Pi_{\bullet} \twoheadrightarrow G_k$, augmented fundamental group of C and X attached to $\mathbb{M}^{\Theta}$, see [IUTChII] Rem. 1.1.1 (iv)-(v).

※ *For the (tempered) Frobenioid version of $\mathbb{M}^{\Theta}$, we refer to [IUTChI] Ex. 3.2 that super-seeds the original [EtTh] §3 - §5. A daring reader will refer to [IUTChIII] Th. 2.2 and Cor. 2.3 for respectively the Frobenioids and Hodge theaters versions of multiradiality and Kummer compatiblity properties. For further details on the cohomological construction of the étale theta function and its anabelian properties, see Talk 3.3.*

Talk 2.3 - Log-Theta Lattice: Symmetries and Indeterminacies. We reach our original goal as stated in Talk 1.3 which is to obtain a meaningful height-comparison in terms of ⊞/⊠-monoids structures after $(-)^{\ell}$-isogeny, or in IUT semantic, *to build a section between some regions of the log-Theta lattice* of Fig. 8 – see "splitting monoids of logarithmic Gaussian procession monoids", Th. A [IUTChIII]. To this end, we consider some *q-pilot and Θ-pilot objects* – some (globally realified) Frobenoids related to F_{mod} and defined at $v \in \mathbb{V}^{bad}$ by the arithmetic line bundle attached respectfully to (q-pilot): the zero locus of $\underline{q}_{\underline{v}}$ in the codomain of the Θ-link; and (Θ-pilot): of the $\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{j=1,\dots,\ell^{*}}$ in the domain of the Θ-link, see [IUTChIII] Def. 3.8. The comparison of their *log-volumes* with respect to certain log-procession and *log-shell* regions provide the estimate.
The creation of the section involves a certain log-theta wandering between $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters, see Fig. 15, that rely on additional multiradial and coric properties of the objects, see Talk 2.2 and below.

※ *Since within the log-theta lattice the Θ-link depends on the ⊠-monoid structure and is incompatible with the ⊞-one, we write $^{\dagger}(-)$ and $^{\ddagger}(-)$ to distinguish different ring-structures induced by the logarithm. Note that both the existence of the section and the meaningfulness of the log-volume comparison are consequences of the symmetries that have been preserved in the previous constructions.*

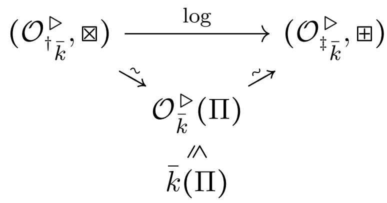
§Three Kinds of Indeterminacies on Log-Shells & Multiradiality. The introduction of mild indeterminacies that reflect the canonical isomorphism classes of objects endows the Θ-link in a global multiradial process or algorithm – see Fig. 14 – that provides **(1)** a section on a region of the log-theta lattice, and **(2)** a log-volume comparison of the q- and Θ-objects – see [Alien] §3.7 (i).

Log-shells. In the context of the Frobenius-étale picture of Fig. 7, we introduce the following *log shell* – see also [Alien] §2.12.3 (iv):

$$\mathcal{I}(\Pi) = \tfrac{1}{2p}\log(\mathcal{O}_{\bar{k}}^{\times}(\Pi)) \leqslant \bar{k}(\Pi)$$

whose property is to fix the non-commutativity of Fig. 7: it has the *upper-semi-compatibility* property which is to contain both $\mathcal{O}^{\triangleright}(\Pi)_k$ and $\log(\mathcal{O}_{\bar{k}}^{\times}(\Pi))$ – i.e. the images of the Kummer morphisms.
Note that since this log-shell can be reconstruct without the ring structure, it is said to be *mono-analytic*.

Fig. 13. *log-link, ⊞/⊠-structures and étale container.*



Log-shells comes indeed in 4 variants – see [Alien] §3.6 (iv) or [IUTChIII] Prop. 1.2 (vi):

(i) *Holomorphic Frobenius-like:* At fixed position (m, n) in the log-theta lattice;
(ii) *Holomorphic étale-like:* Vertical coric $(n, \bullet)$, related to $\bar{k}(\Pi)$;
(iii) *Mono-analytic Frobenius-like:* At fixed position (m, n), related to $\mathcal{O}_{\bar{k}}^{\times\mu}$;
(iv) *Mono-analytic étale-like:* Bicoric $(\bullet, \bullet)$, related to $\mathcal{O}_k(G)$.

Note that all of the variant will be required for the correct estimate.

The multiradial algorithm then goes as in Fig. 14 by producing from non-vertical or non-horizontal invariants strips some multiradial objects.

Fig. 14. *Multiradial algorithm: Kummer and forget, from $\mathcal{F}$ to $\mathcal{D}^{\vdash}$-prime strip.*

Arithmetic Holomorphic Frobenius	$\overset{Kummer}{\rightsquigarrow}$	Arithmetic Holomorphic Étale	$\overset{Forget}{\rightsquigarrow}$	Mono-analytic Étale
$\mathcal{F}$-prime strips		*$\mathcal{D}$-prime strip*		*$\mathcal{D}^{\vdash}$-prime strips*

Indeterminacies. We briefly review the log-shell mild indeterminacies:

(Ind1) At the $\mathcal{D}$-étale level of Hodge theater, the Θ-link is defined as full polymorphisms ${}^{\dagger}\Pi \twoheadrightarrow {}^{\dagger}G \overset{full\ poly}{\simeq} {}^{\ddagger}G \twoheadleftarrow {}^{\ddagger}\Pi$. This indeterminacy arises from the $\mathcal{D}^{\vdash}$-prime strip to synchronize theses symmetries with the more rigid Frobenius objects that they contains via Kummer;

(Ind2) This indeterminacy arises from the $\mathcal{F}^{\vdash\times\mu}$-prime strip and the $\widehat{\mathbb{Z}}^{\times}$-indeterminacy acting on $\mathcal{O}^{\times\mu}$;

(Ind3) This indeterminacy reflects the upper-semi-compatibility of the log-shells.

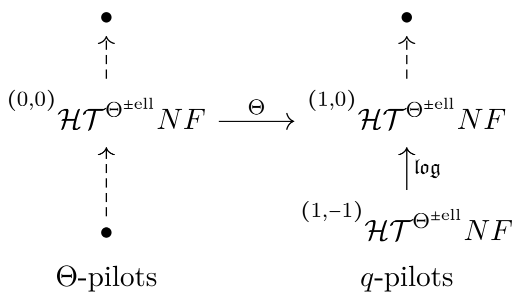
Note that these indeterminacies are formulated at the level of *tensor packets attached to procession* $\mathbb{S}$ which achieve multiradiality at the level of labels in $\mathbb{F}_{\ell}^{*}$ – see [Alien] §3.6 (v). (Ind1) and (Ind2) comes from internal symmetries of the $\Theta^{\pm\text{ell}}$NF-Hodge theaters, while (Ind3) is a gobal symmetry on the vertical strip of the log-theta lattice.

§The Main Multiradial Algorithm: Log-theta Wandering. The $\boxplus/\boxtimes$-monoid structures comparison process is dealt with by considering an initial q-object in the $\Theta^{\pm\text{ell}}$NF-Hodge theater ${}^{(1,0)}\mathcal{HT}^{\Theta^{\pm\text{ell}}}NF$ as in Fig. 15. On the one hand, the Θ-link being defined with respect to the $\boxtimes$-monoid structure only – as supported by the value group in $\mathcal{O}^{\triangleright}$, this requires the consideration of the

$$\mathfrak{log}\colon {}^{(1,-1)}\mathcal{HT}^{\Theta^{\pm\text{ell}}}NF \to {}^{(1,0)}\mathcal{HT}^{\Theta^{\pm\text{ell}}}NF$$

which intertwins the $\boxplus$ and $\boxtimes$ structures at the level of the $(-)^{\Vdash\blacktriangleright}$ and $(-)^{\vdash\times\mu}$-prime strip of the codomain of Θ.

Fig. 15. *Log-Theta Wandering: two computations of $\log-vol(q)$. $\neq$ column $\neq$ arithm. hol. container.*



On the other hand, the Θ-link acts at the level of the $\mathcal{F}$-prime strips as a gluing isomorphism between the arithmetic holomorphic structures of the two vertical lines. One can thus consider the whole vertical $\mathfrak{log}$-column ${}^{(0,\bullet)}\mathcal{HT}^{\Theta^{\pm\text{ell}}}NF$ in its *$\mathcal{D}^{\vdash}$-mono-analytic étale version* for identifying up to the 3 indeterminacies the required volume estimate:

$$\text{log-vol}[\Theta\text{-object mod (Ind1), (Ind2) and (Ind3)}] \geqslant \text{log-vol}[q\text{-pilot object}]$$

– see [Alien] §3.7 (ii) for a detailed description of the 12-steps estimate process.

※ *The final step for the abc-estimate as in [GenEll] of Talk 1.3 relies on the construction of suitable initial Θ-data as in [IUTChIV] Cor. 2.2, that once the Log-volumes estimates of Θ-pilot objects obtained as in ibid. Th. 1.10, provide as application of IUT the searched Diophantine inequalities of [EtTh] Lem. 3.5 – see [IUTChIV] Cor. 2.3.*

Tab. 3. *Frobenoids components of a theater $\mathcal{HT}^{\Theta^{\pm ell}NF}$: Θ-link properties and cyclotomic rigidity origin; Definition of q-pilot and Θ-pilot objects.*

	Places	Definition	$\{q,\Theta\}$-data	Θ-link	Cyclotomic Rigidity
Local unit group	$\mathbb{V}^{non}$	$G_k \curvearrowright \mathcal{O}_{\bar{k}}^{\times\mu}$ and $\{\mathcal{O}_{\bar{k}^H}^{\times\mu} \leqslant (\mathcal{O}_{\bar{k}}^{\times\mu})^H\}_{H \leqslant^{op} G_k}$ With $k = K_v$	$a^{\square}$	iso	Brauer group
Local value group	$\mathbb{V}^{bad}$	$\langle \underline{\underline{q}}_v^{j^2},\ j = 1,\dots,l^{*}\rangle \overset{Mon.}{\simeq} \mathbb{N} \leqslant \mathcal{O}_{K_{\underline{v}}}^{\triangleright}$ With $\underline{\underline{q}}_v \in \mathcal{O}_{K_{\underline{v}}}^{\triangleright}$ a 2ℓth root of $\underline{q}_v$, Tate param. of E_K	$b^{\square}$	dilate	Mono-theta env.
Global value group	$\mathbb{V}^{bad}$	Realified Frobenoid	$c^{\square}$	dilate	κ-coric rat. func.
$\{q,\Theta\}$***-Pilot object***		A (global) $c^{\square}$-data defined by the local $b^{\square}$ at $\underline{v} \in \mathbb{V}^{bad}$		–	–

References.

[Moc03b] S. Mochizuki, "Topics surrounding the anabelian geometry of hyperbolic curves," in *Galois groups and fundamental groups*, ser. Math. Sci. Res. Inst. Publ. Vol. 41, Cambridge Univ. Press, Cambridge, 2003, pp. 119–165, MR2012215, Eprint available on-line.

[EtTh] ——, "The étale theta function and its Frobenioid-theoretic manifestations," *Publ. Res. Inst. Math. Sci.*, vol. 45, no. 1, pp. 227–349, 2009, MR2512782, Eprint available on-line.

[Alien] ——, "The mathematics of mutually alien copies: From Gaussian integrals to Inter-universal Teichmüller theory," *RIMS Preprint* ***no. 1854***, 169p. Jul. 2016, Eprint available on-line.

[Mum72] D. Mumford, "An analytic construction of degenerating curves over complete local rings," *Compositio Math.*, vol. 24, pp. 129–174, 1972, MR352105.

Topic 3 - Anabelian Geometry

The goal of this section is to present the reconstruction results of *absolute mono-anabelian geometry* on which relies the IUT geometry for the variation of the $(\boxplus, \boxtimes)$-Ring structures – see more especially Talk 2.2. Recall that anabelian geometry deals with the Grothendieck Conjecture (GC), i.e. the question of reconstructing isomorphism classes of schemes from combinatorial and profinite group theoretic data – that is in its *relative and bi-anabelian form*, of the bijectivity of $\mathrm{Hom}_k(X,Y) \to \mathrm{Out}_{G_k}(\Pi_X, \Pi_Y)$ for X, Y given schemes over k, and where $\Pi_\bullet$ denotes the étale fundamental group.
A first introductory Talk 3.1 presents the seminal relative bi-anabelian Theorem A of Mochizuki [pGC] that provides a positive result to GC for hyperbolic curves over sub-p-adic fields. Following the IUT's consideration of smooth curves with semistable singular special fibre over non-archimedean fields, Talk 3.2 presents the basis of André's tempered fundamental group for rigid analytic spaces [And03], the main *combinatorial anabelian* reconstruction results for curves – see [SemiAnbd] Cor. 3.11, as well as its profinite compatibility – see [IUTChI] Cor. 2.5. Talk 3.3 focuses on the tool of Elliptic and Belyi cuspidalization of [AbsTopII] that allows, by adding an inertia data to Mochizuki's Th. A certain anabelian reconstructions **(1)** of function and constant fields, and **(2)** of theta functions and mono-theta environments [AbsTopIII] & [EtTh] to obtain the rigidity properties of Talk 2.2.

※ *These anabelian constructions provide, via the properties of Talk 2.2, the core of the $\square$-prime strips – e.g. for $\square \in \{\mathcal{F}^{\Vdash \blacktriangleright \times \mu}, \mathcal{D}, \mathcal{D}^{\vdash}\}$ – then of the $\Theta^{\pm ell}$NF-Hodge theaters and their $\mathcal{D}^{\vdash}$-bases as in Talk 2.1 and 2.3 and in Fig. 14. At an elementary anabelian level, it is well-known that non-isomorphic p-adic local fields can have isomorphic Galois groups; an anabelian defect that can be solved by fixing some additional data – such as the ramification filtration – whose role can be seen as rigidifying the fluidity of the $\boxplus$-monoidal structure. We refer to [Jos20] for illustrations of this principle for Galois realisations and elliptic curves, and in terms of IUT, to §1.6 ibid with respect to the Ind1 indeterminacy, and to [Jos19b] ibid in terms of some universal addition law. See also [Hos16] Rem. 4.3.3 for similar statements more broady directed towards IUT.*

Talk 3.1 - Relative Bi-Anabelian Geometry. For X algebraic variety over a field K, one can form from the relative morphism $X \to \mathrm{Spec}\ K$ a Fundamental Exact Sequence:

$$1 \to \Delta_X \to \Pi_X \to \mathrm{Gal}_K \to 1, \qquad \mathrm{Hom}_K^{\mathrm{dom}}(X,Y) \to \mathrm{Hom}^{\mathrm{open}}_{\mathrm{Gal}_K}(\Pi_X, \Pi_Y)_{\Delta_Y} \qquad \text{(FES/Hom)}$$

where Π_X denotes the étale fundamental group $\pi_1^{et}(X, *)$ and $\Delta_X < \Pi_X$ denotes the kernel of the projection $pr_X \colon \Pi_X \to Gal_K$, isomorphic to the geometric $\pi_1^{et}(X \otimes \bar{K}, *)$. Relative bi-anabelian geometry deals with the question of reconstructing morphisms $X \to_K Y$ over Spec K from Gal_K-equivariant morphisms $\Pi_X \to \Pi_Y$ up to Δ_Y-inner automorphisms – i.e. the bijectivity of the RHS in (FES/Hom). We refer to [pGC] and to [NTM98] for a general presentation.

§GC for Hyperbolic Curves. After some seminal work of Nakamura and Tamagawa, a fundamental breakthrough was achieved with Mochizuki Theorem A [pGC] that shifts to p-adic analytic techniques over *local fields* and establishes *the bijectivity of* (FES/Hom) *for smooth hyperbolic curves over sub-p-adic field K.*

Very roughly the key points of the approach rely on p-adic Hodge theory and are as follows – see [pGC]:

(i) *Global differentials as container.* Writing $D_X = H^0(X_K, \omega_{X/K})$, the differentials provide a potential container $X_K \hookrightarrow \mathbb{P}(D_X)$, where indeed $D_X \simeq (\pi_1^{(p)}(X_{\bar{K}})^{ab} \otimes \mathbb{C}_p(1))^{\mathrm{Gal}_K}$;

(ii) *Arithmetic line bundle.* From the existence of an L-arithmetic line bundle on Y^H via its Chern class follows the existence of L-rational point(s) x_L^H in Y^H for every $H <^{op.} \pi_1^{(p)}(X_K)$;

(iii) *A geometric α_L:* $\mathrm{Gal}_L \to \pi_1^{(p)}(X)$. Considering a certain small p-adic open set $\mathrm{Spec}\, L$ in X_K, an argument *à la* Tamagawa provides the p-adic convergence of $x_L^H \underset{H}{\to} x_L^\infty \in X(L)$ that induces α_L, and which in turns by Falting's p-adic Hodge theory produces Φ_α: $\mathrm{Spec}\ L \to \mathbb{P}(D_X)$ over K whose (closure of the) image is X_K.

We once more refer to [Fal98] and [NTM98] §5.2 for an overview of the proof and details.

§Towards Mono-Anabelian Results. Having in mind the goal of building some IUT mono-anabelian functorial group-theoretic algorithms, note that for and over K a p-local field, the following data can indeed be turn-by-turn *group-theoretically* reconstructed from G_K, resp. Π_X:

(i) *Fields.* **(i)** the data p, then $[K:\mathbb{Q}_p]$ and $e(K)$ by local class field theory via $G_K^{ab} \simeq \widehat{K}^\times$ and its rank, **(ii)** then I_K, P_K and the Frobenius $Frob_K$ via p-log and open sub-groups of G_K, then the multiplicative ring $K^\times$, **(iii)** then $\bar{K}^\times$ and $\mu_{\mathbb{Q}/\mathbb{Z}}(\bar{K})$ by Verlangerung, and **(iv)** the isomorphism $H^1(Gal_K, \mu_{\mathbb{Q}/\mathbb{Z}}(\bar{K})) \simeq \mathbb{Q}/\mathbb{Z}$ – see [AbsAnAb] Prop. 1.2.1 and its proof;

(ii) *Hyperbolic Curves of type (g,r).* **(i)** the kernel $\Delta < \pi_1(X, \bar{x})$ by Tamagawa semi-abelian techniques, and **(ii)** the data (g,r) via $Frob_k$ and $\dim(\Delta \otimes \mathbb{Q}_\ell)$ – see [AbsAnAb].

※ *By providing some additional inertia data such as given by Belyi/Elliptic cuspidalization, this relative anabelian result will turn into an absolute one – cf Talk.3.3. In higher dimension, typical anabelian results are established* ***(1)*** *for configuration spaces of hyperbolic curves first over finitely generated fields of characteristic* 0 *[Mochizuki, Nakamura, Takao & Tamagawa] then over sub-p-adic fields [Mochizuki, Hoshi, Minamide], and more recently* ***(2)*** *for any fundamental system of Zariski neighourhoods of smooth varieties over a number field via étale homotopy type [Schmidt, Stix 2016], and* ***(3)*** *idem but for polycurves over any sub-p-adic field [Hoshi 2018].*

TALK 3.2 - TEMPERED ANABELIAN GEOMETRY. IUT deals with the rigid analytic geometry of elliptic curves $E(k_v)^{rig} \simeq \mathbb{G}_m^{rig}/q_v^{\mathbb{Z}}$ and their symmetries via the local Tate parameter $q_v \in k_v$ at places of bad multiplicative reduction – alt. the values of a certain étale p-adic theta function, see Talk 2.2. The anabelian context is here provided by André's *tempered fundamental group* $\pi_1^{\mathrm{temp}}(X,x)$ – see [And03] §4-5 for an introduction and [SemiAnbd] Ex. 3.10 for a reformulation in terms of log-smooth curves – that is **(1)** functorial, **(2)** provides discrete covers over singular base *à la* SGA3, comes with a (FES/Hom), and **(3)** is étale compatible – e.g. $\widehat{\pi}_1^{\mathrm{temp}}(X) \simeq \pi_1^{et}(X)$.

§Combinatorial Tempered Anabelian Geometry. Anabelian principles are those of *combinatorial anabelian geometry*, that encodes the anabelian properties – $\pi_1^{\mathrm{temp}}(X)$, decomposition and inertia groups – of smooth pointed hyperbolic curves in certain *semigraphs* (or temperoids $\mathcal{G}$) of branches, vertices and edges representing the dual semi-graph of one of their semistable models – see [Lep10] §1.2 for an introduction.

In this context the two key reconstruction results for log-smooth curve over p-adic local field are:

(i) 2-*classes of semigraphs of anabelioids are recovered via isomorphisms of their tempered fundamental groups*, [SemiAnbd] Cor. 3.11;

(ii) *the verticial subgroups (resp. edge-like subgroups) of* $\pi_1^{\text{temp}}(\mathcal{G})$ *are exactly the maximal compact subgroups of* $\pi_1^{\text{temp}}(\mathcal{G})$ *(resp. non-trivial intersection of)*, ibid. Th. 3.7 (iv)

i.e. combinatorial anabelian geometry has no "fake inertial/decomposition groups".

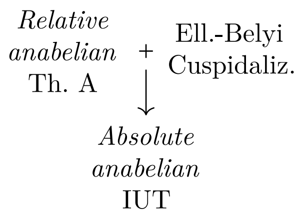
§Tempered & Pro-conjugate. Considering $\Pi_X^{\text{temp}} \hookrightarrow \Pi_X^{et}$, the following result allows in IUT the synchronisation of the $\{\pm 1\}$-indeterminacies associated to various valuations in $\Theta^{\pm \text{ell}}$NF-Hodge theaters in Fig. 5 – see [IUTChI] Cor. 2.5:

(i) an inertia subgroup of $\widehat{\Pi}_X$ associated to a cusp of X is in Π_X^{temp} if and only if it is an inertia subgroup of Π_X^{temp};

(ii) the only $\widehat{\Pi}_X$-conjugate of Π_X^{temp} containing an inertia group of Π_X^{temp} is Π_X^{temp} itself.

Following ibid. Rem. 2.5.1 this result can be seen as a certain "relative tempered-profinite" Section Conjecture.

TALK 3.3 - IUT ABSOLUTE MONO-ANABELIAN RECONSTRUCTIONS. As encountered in Talk 2.2, anabelian reconstructions in IUT are essentially *absolute and mono-anabelian* – i.e. deal, for a given topological group Π, abstractly isomorphic to some Π_X, with the reconstruction of X by a group-theoretic algorithm. Over k MLF, they are of 2 types: **(1)** with respect to the function field K_X, and **(2)** with respect to étale theta-functions of elliptic orbicurves. In these situations, a key obstruction to anabelianity lays in the lack of terminal object – or k-core – in a certain category $\overline{\text{Loc}}_k(X)$ of finite étale covers, see [AbsTopII] Def. 3.1 & [Moc03a] §2.



The consideration of hyperbolic orbicurves that are of *strict Belyi types or elliptically admissible* resolves this obstruction. By providing some additional inertia group data to Mochizuki Th. A of Talk 3.1 in terms of Belyi/elliptic cuspidalization – see [AbsTopIII] Rem. 1.11.1 (i), it provides first some intermediate anabelian reconstructions, which in turns recovers by Kummer theory our objects in some anabelianly rebuilt *étale containers* – e.g. $\bar{k}^\times \hookrightarrow H^1(G(\Pi), \Lambda(\Pi))$, $\Gamma(U, \mathcal{O}_U^\times) \hookrightarrow H^1(\pi_U, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\pi_X))$, or $\{\underline{\mathbb{Z}} - orbit\ of\ \text{étale}\ theta\ functions\} \hookrightarrow H^1(\underline{\Pi}_{\ddot{Y}}^{\text{temp}}, \underline{\Delta}_\Theta)$ below.

※ *Mono-anabelian obstacle for MLF: there exists* (k_1, k_2) *MLFs such that* $G_{k_1} \simeq G_{k_2}$ *and* $k_1 \not\simeq k_2$*. For case (i) ibid. i.e. the mono-anabelian reconstruction for a given* $\Pi \curvearrowright M$ *of the* $\{\pm 1\}$*-orbits (resp.* $\widehat{\mathbb{Z}}^\times$*-orbits) of* $\mu_{\widehat{\mathbb{Z}}}(M) \xrightarrow{\sim} \mu(\Pi)$ *for* $(\Pi \curvearrowright M) \simeq (G_k \curvearrowright \bar{k}^\times)$ *(resp.* $(\Pi \curvearrowright M) \simeq (G_k \curvearrowright \mathcal{O}_{\bar{k}}^\times)$*) via LCFT [AbsTopIII] Prop. 3.3 (i).*

§Hidden Symmetries and Elliptic-Belyi Cuspidalisation. The following notions for hyperbolic orbicurves guarantee the reconstruction of decomposition groups of closed points of X in Π_X, which is the first step in the reconstruction of K_X.

A hyperbolic orbicurve X over k of characteristic 0 is said to be:

(i) *elliptically admissible* if it admits a k-core $X \to C$ where C is the stack-quotient $C \simeq E \smallsetminus \{0\}//\{\pm 1\}$ for some elliptic curve E over k;

(ii) *of strict Belyi type* if defined over a NF and isogenous to a genus 0 curve.

In the first case, denoting $E^\times = E \smallsetminus \{0\}$, $E_N = E \smallsetminus E[N]$, $U_{C,N} = E_N//\{\pm 1\}$, and $U_{X,N}$ base change as in [Yam17] §3.2.1 – Fig. 16 illustrates how the hidden symmetries of N-torsion points of E provides some open immersions $U_{X,N} \hookrightarrow X$ as finite étale morphisms [AbsTopII] Def. 3.1; A similar context is provided by orbicurves of strict Belyi types, see Fig. 17, where the left column is given by definition and the right one follows the existence of Belyi maps, [AbsTopII] Ex. 3.2.

Fig. 16. *Elliptic Hidden Symmetries.*

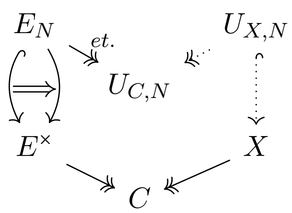
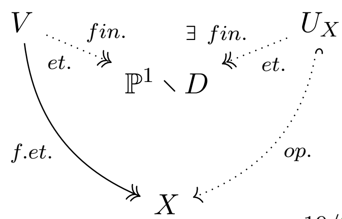


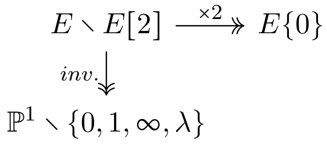
Fig. 17. *Belyi Cuspidalization - up to base field extension.*



In both cases, for X and $U_{X,(N)}$ given, *one recovers from* $\Delta_X < \Pi_X$ *and from the Belyi (resp. Elliptic) Cuspidalizations* $\{\Pi_{U_X} \to \Pi_X\}_{U_X \hookrightarrow X}$ *(resp.* $\{\Pi_{U_{X,N}} \to \Pi_X\}_{U_{X,N} \hookrightarrow X}$*) the set of decompositions groups* at the points of $X \smallsetminus U_{X,N}$, see [AbsTopII] Cor. 3.7 (resp. [AbsTopIII] Cor. 3.3).

※ *Note* ***(1)*** *that the existence of a Belyi map from a pointed elliptic curve to* $\mathbb{P}^1 \smallsetminus \{0,1,\infty\}$ *presents any elliptically admissible orbicurve over a NF as being of strict Belyi type [AbsTopIII] Rem. 2.8.3, and* ***(2)*** *that similar constructions do not hold for* $g > 1$*. We refer to [AbsTopII] Cor. 3.3 & 3.7 (resp. Rem. 3.3.3 & 3.7.1) for further anabelian reconstructions with respect to elliptical & Belyi type curves (resp. for* Π_X^{temp}*).*

Fig. 18. *Elliptic to Belyi.*



§Absolute Mono-anabelian Reconstructions (1). Let us briefly present step by step how the strict Belyi type provides the anabelian reconstruction over MLF of the function field K_X and of the base field k, see [AbsTopIII] Th 1.9. We denote by $S < X$ a finite set of closed points, write $U = X \smallsetminus S$ and equivalently M_X or $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\pi_X)$.

(i) *Inertia and decomposition groups.* Belyi cuspidalization reconstructs respectively the decomposition groups D_x then the inertia groups $I_x = D_x \cap \Delta_U$ of $I_x < \pi_1(X \smallsetminus U)$;

(ii) *Synchronization of geometric cyclotomes.* On obtains a cyclotomic rigidity isomorphism, i.e. a canonical identification $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\pi_U) \simeq I_x \simeq \widehat{\mathbb{Z}}(1)$ of for every $x \in U$, see [AbsTopIII] Prop. 1.4;

(iii) *Multiplicative group* $K_X^\times$ *and* $k^\times$. The reconstruction of principal divisors and the Kummer map $\kappa_U : \Gamma(U, \mathcal{O}_U^\times) \hookrightarrow H^1(\pi_U, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\pi_X))$ provide the reconstruction of $\bar{k}^\times < K_{X,\bar{k}} \hookrightarrow \lim_{S,k'} H^1(\pi_{X\smallsetminus S}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\pi_U))$; ibid Prop. 1.6 & 1.8;

(iv) *Fields* K_X *and* k. By Uchida's Lemma ibid. Prop. 1.3, the additive structure of $K_X^\times \cup \{0\}$ is recovered from the set of valuations and from $K_X^\times$.

These constructions steps indeed first go through the ones of the number field part k_{NF} of k, and rely on some additional similar constructions with respect to the maximal abelian cuspidalization $\Pi_U^{c\text{-}ab}$ of Π_U, and the Kummer $k^\times \hookrightarrow H^1(G_k, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\pi_X)) \hookrightarrow H^1(\pi_X, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\pi_X))$ – see ibid. Cor. 1.10 (ii) (d) and (iii) (e)-(h). Note also the synchronization of cyclotomes in (ii) which allows to reconstruct a *container* $H^1(\Pi, M) \simeq H^1(\Pi_X, M_X)$.

§Absolute Mono-anabelian Reconstructions (2). We now deal with the anabelian reconstruction of the *étale theta function and Mono-theta environments* with respect to the tempered fundametal group of Talk 3.2 and [SemiAnbd]. Let thus k be a p-adic local field – ie $[k : \mathbb{Q}_p] < \infty$ –, and X/k be an elliptic curve with split multiplicative reduction over $\mathcal{O}_k$.

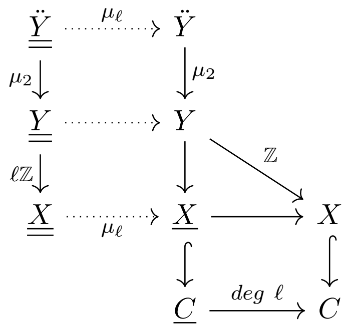
Let us assume given Π a fixed topological group – abstractly corresponding to Π_X^{temp}. Then some functorial group-theoretic algorithm recovers:

(i) *Coverings.* The various covers $\{\ddot{Y}, Y, X, \underline{\underline{\ddot{Y}}}, \underline{\underline{Y}}, \underline{\underline{X}}, \dots\}$ involved in Fig. 12 & 19, see [EtTh] Prop. 2.4;

(ii) *Étale Theta Functions.* The topological groups $G(\Pi)$ and Δ_Θ, and thus a certain subset of cohomological classes in the container $H^1(\underline{\Pi}_{\ddot{Y}}^{\text{temp}}, \underline{\Delta}_\Theta)$ that corresponds to *the* $\underline{\mathbb{Z}}$*-orbit of the étale theta function* $\mathcal{O}_{\ddot{K}}^\times . \ddot{\eta}^\Theta(\Pi)$, see ibid. Th 1.6;

(iii) *Mono-theta environement* $\mathbb{M}^\Theta(\Pi)$. See ibid. Cor. 2.18 (ii) & (iii);

Fig. 19. *Covers for* Θ *and* $\Theta^{1/\ell}$.



Note that these steps also recover, via the cohomological class $\underline{\underline{\ddot{\eta}}}^\Theta \in H^1(\underline{\Pi}_{\underline{\underline{\ddot{Y}}}}^{\text{temp}}, \ell.\underline{\Delta}_\Theta)$, a ℓ-root of the étale theta function, see Fig. above.

※ *The constant multiple rigidity of Talk 2.2* §The mono-theta environment *follows from elliptic cuspidalization, see [EtTh] Cor. 2.19 (iii), which moreover gives reconstruction results for Archimedean places [AbsTopIII] Cor. 2.7 & 2.9. For mono-anabelian reconstructions over MLF and of "MLF pair"* $G \curvearrowright M$*, we also refer to the synthetic [Hos16] – esp. Summaries 3.15 & 4.3.*

References.

[And03] Y. André, "On a geometric description of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and a p-adic avatar of $\widehat{GT}$," *Duke Math. J.*, vol. 119, no. 1, pp. 1–39, 2003, MR1991645.

[Fal98] G. Faltings, "Curves and their fundamental groups (following Grothendieck, Tamagawa and Mochizuki)," in 252, Séminaire Bourbaki. Vol. 1997/98, 1998, Exp. No. 840, 4, 131–150, MR1685577.

[Hos16] Y. Hoshi, "Introduction to mono-anabelian geometry," in *Proceedings of the conference "Fundamental Groups in Arithmetic Geometry"*, Paris, France, 2016, Eprint available on-line.

[Jos20] K. Joshi, "On Mochizuki's idea of Anabelomorphy and its applications," *arXiv e-prints*, Mar. 2020. arXiv: 2003.01890.

[Lep10] E. Lepage, "Tempered fundamental group and metric graph of a Mumford curve," *Publ. Res. Inst. Math. Sci.*, vol. 46, no. 4, pp. 849–897, 2010, MR2791009.

[pGC] S. Mochizuki, "The local pro-p anabelian geometry of curves," *Invent. Math.*, vol. 138, no. 2, pp. 319–423, 1999, MR1720187, Eprint available on-line.

[Moc03a] ——, "The absolute anabelian geometry of canonical curves, Kazuya Kato's fiftieth birthday," *Doc. Math.*, vol. Extr. Vol. Pp. 609–640, 2003, MR2046610, Eprint available on-line.

[AbsAnAb] ——, "The absolute anabelian geometry of hyperbolic curves," in *Galois Theory and Modular Forms*, K. Hashimoto, K. Miyake, and H. Nakamura, Eds. Kluwer Acad. Publ., Boston, MA, 2004, vol. 11, pp. 77–122, MR2059759, Eprint available on-line.

[SemiAnbd] ——, "Semi-graphs of anabelioids," *Publ. Res. Inst. Math. Sci.*, vol. 42, no. 1, pp. 221–322, 2006, MR2215441, Eprint available on-line.

[EtTh] ——, "The étale theta function and its Frobenioid-theoretic manifestations," *Publ. Res. Inst. Math. Sci.*, vol. 45, no. 1, pp. 227–349, 2009, MR2512782, Eprint available on-line.

[AbsTopII] ——, "Topics in absolute anabelian geometry II: Decomposition groups and endomorphisms," *J. Math. Sci. Univ. Tokyo*, vol. 20, no. 2, pp. 171–269, 2013, MR3154380, Eprint available on-line.

[AbsTopIII] ——, "Topics in absolute anabelian geometry III: Global reconstruction algorithms," *J. Math. Sci. Univ. Tokyo*, vol. 22, no. 4, pp. 939–1156, 2015, MR3445958, Eprint available on-line.

[NTM98] H. Nakamura, A. Tamagawa, and S. Mochizuki, "Grothendieck's conjectures concerning fundamental groups of algebraic curves," *Sūgaku*, vol. 50, no. 2, pp. 113–129, 1998, (Eng. Sugaku Expositions 14, AMS (2001), no. 1), MR1648427.

Topic 4 - Advanced Talks

ATA - Mono-anabelian Transport in Inter-universal Teichmüller Theory — Y. Hoshi

Abstract: In this talk, I first give a brief review of the content of the main theorem of inter-universal Teichmuller theory from the point of view of mono-anabelian transport. After the review, I also explain the relationship between the main theorem and an inequality of log-volumes.

ATB - Explicit Estimates in Inter-universal Teichmüller Theory — A. Minamide

Abstract: In the final paper of a series of papers concerning inter-universal Teichmuller theory, Mochizuki verified various numerically non-effective versions of the Vojta, ABC, and Szpiro Conjectures over number fields. In this talk, we will give various numerically effective versions of Mochizuki's results. This is joint work with S. Mochizuki, I. Fesenko, Y. Hoshi, and W. Porowski.

ATC - An Introduction to p-adic Teichmüller Theory — Y. Wakabayashi

Abstract: This talk aims to give an introductory exposition of p-adic Teichmüller theory. In a series of papers on IUT, S. Mochizuki refers that theory from the viewpoint of the analogy with IUT. Relative to this analogy, (one-punctured) elliptic curves over a number field correspond to "nilpotent ordinary indigenous bundles" over a hyperbolic curve in positive characteristic. Nilpotent ordinary indigenous bundles play essential roles in p-adic Teichmüller theory because they are used to construct p-adic canonical liftings of the underlying curves. In this talk, I would like to talk about these objects and related topics from the beginning.

Interactive Q&A Session on the Essential Logical Structure of Inter-universal Teichmüller Theory
S. Mochizuki

Abstract: Introductory lectures and expositions on inter-universal Teichmüller theory — such as, for instance, [Alien] — have a tendency to concentrate on exposing the technical details surrounding the various mathematical objects that appear in the theory. To a certain extent, of course, this is unavoidable. On the other hand, concentrating on such technical details can lead to a situation where one is overwhelmed with seemingly meaningless technicalities to such an extent that one loses sight of the *essential logical structure* of the theory.
The purpose of this session will be to discuss, in as interactive a fashion as is possible, this *essential logical structure* of the theory, as exposed in [EssLgcIUT] (cf., especially, §3.3, §3.10, §3.11). The discussion will center around the following topics (not necessarily in the following order):
(T1) logical AND "$\wedge$" versus logical OR "$\vee$" and the use of distinct labels,
(T2) dilated versus nondilated dimensions and the analogy with classical complex Teichmüller theory,
(T3) the theta-OR indeterminacy ("(ΘORInd)") versus the log-OR indeterminacy ("(logORInd)"),
(T4) the symmetries/nonsymmetries and coricities of the Frobenius-like/étale-like portions of the log-theta-lattice,
(T5) the Kummer theory relating Frobenius-like and étale-like structures and the resulting Kummer-detachment indeterminacies,
(T6) the significance of the theta function and the analogy with the Jacobi identity of the theta function,
(T7) the central importance of the log-Kummer-correspondence (and Galois evaluation) and the (Ind3) indeterminacy,
(T8) the importance of simulating — via the combinatorial structure of a Hodge theater — a global multiplicative subspace (GMS) and global canonical generator (GCG),
(T9) the importance of conjugate synchronization and the relationship with the Kummer theory of the theta function.

Organization and Schedule

The seminar takes place *every two weeks* for ~ 2 hours on Thursday by `Zoom` meeting between 17:30-19:30, JP time – 6:30pm and 8:30pm during the European winter time November-March – i.e. 9:30-11:30, UK time, and 10:30-12:30 FR time.

Website of the Seminar.

Note that in order to ensure a cohesive and focused working group, participation is restricted to a limited number of participants, and that **registration is done by invitation only.**

Speakers will give their presentation either:
(a) by video conferencing and black board,
(b) by a `Zoom` shared whiteboard via (graphic) tablet.

The **audience** will attend the talk (a') globally via a screening with video projector or (b') individually via `Zoom`. Questions will be submitted to the speaker either by chat or verbally.

A temporary private `Slack` group chat **"[RIMS-Lille] IUT"** has been setup for sharing resources, questions and informal discussions – participants are automatically invited.

The website of the seminar with updated schedule, list of participants, and references can be found following the above QR-clickable link.

※ *We refer to RIMS - Homotopical Anabelian Geometry Seminar and for RIMS - Homotopical Arithmetic Geometry Seminar - 7 Topics for software and hardware references.*

Organizational Meeting (Talk 0 on Sept. 24, 2020). An organization meeting will take place for giving a complete overview of this guide and for distributing-confirming the speakers and slots talk.

Talks and Speakers. If required speakers can use a double or a shared slot for their talk.

September				
0.	24th	T0	Collas	*RIMS - Japan*
October				
1.	8th	T1.1	Cluckers – Fresse	*Lille - France*
2.	29th	T3.1	Porowski	*Nottingham - UK*
November				
3.	5th	T1.2*	Dèbes	*Lille - France*
4.	19th	T3.2	Tsujimura	*RIMS - Japan*
December				
5.	3rd	T1.3	Liu	*Bordeaux - France*
6.	17th	T2.1	Minamide	*RIMS - Japan*

January				
7.	21th	T2.2	Porowski	*Nottingham - UK*
February				
8.	4th	T3.3	Sawada	*Osaka - Japan*
9.	18th	T2.3	Minamide	*RIMS - Japan*
March				
10.	18th	Q&A	Mochizuki	*RIMS - Japan*
11.	25th	ATA	Hoshi	*RIMS - Japan*
April				
12.	8th	ATB	Minamide	*RIMS - Japan*
13.	22th	ATC	Wakabayashi	*Tokyo - Japan*

The order of talks will be adapted following the availability of the speakers. (*) Mochizuki at Berkeley Colloquium 4:10pm to 5:00pm (PST-US) – i.e 1:10 to 2:00 (CET-FR); 9:10am to 10:00am (JST-JP).

Advanced Talks. Following the participants, some dedicated talks are given by experts of the field on **(ATA)** some absolute mono-anabelian geometry aspects, **(ATB)** Recent developments and application of IUT, and **(ATC)** p-adic Teichmüller Theory.

List of Participants (36). Participants who have so far declared their interest in attending or giving a talk.

(i) Seguin Béranger, Lille University, France;
(ii) Niels Borne, Lille University, France;
(iii) Raf Cluckers, CNRS Lille University, France & KU Leuven, Belgium;
(iv) Benjamin Collas, RIMS - Kyoto University, Japan;
(v) Weronika Czerniawska, University of Geneva, Switzerland;
(vi) Pierre Dèbes, Lille University, France;
(vii) Ivan Fesenko, Nottingham University, UK;
(viii) Benoit Fresse, Lille University, France;
(ix) Yuta Hatasa, Tokyo Institute of Technology, Japan;
(x) Julien Hauseux, Lille University, France;
(xi) Watanabe Hiroyuki, RIMS - Kyoto University, Japan;
(xii) Yuichiro Hoshi, RIMS - Kyoto University, Japan;
(xiii) Angelo Iadarola, Lille University, France;
(xiv) Shun Ishii, RIMS - Kyoto University, Japan;
(xv) Fumiharu Kato, Tokyo Institute of Technology, Japan;
(xvi) Qing Liu, Bordeaux University, France;
(xvii) Arata Minamide, RIMS - Kyoto University, Japan;
(xviii) Shinichi Mochizuki, RIMS - Kyoto, Japan;
(xix) Katharina Müller, University Göttingen, Germany;
(xx) Wojciech Porowski, Nottingham University, UK:
(xxi) Lorenzo Ramero, Lille University, France;
(xxii) Koichiro Sawada, Osaka University, Japan;
(xxiii) Ryoji Shimizu, RIMS - Kyoto University, Japan;
(xxiv) Masatoshi Suzuki, Tokyo Institute of Technology, Japan;
(xxv) Christian Tafula Santos, Université de Montréal, Canada;
(xxvi) Yuichiro Taguchi, Tokyo Institute of Technology, Japan;
(xxvii) Shota Tsujimura, RIMS - Kyoto University, Japan;
(xxviii) Yasuhiro Wakabayashi, Tokyo Institute of Technology, Japan;
(xxix) Naganori Yamaguchi, RIMS - Kyoto University, Japan;
(xxx) Yu Yang, RIMS - Kyoto University, Japan;
(xxxi) Seidai Yasuda, Osaka University, Japan;
(xxxii) Yu Yasufuku, Nihon University, Japan;
(xxxiii) Shigetoshi Yokoyama, Gunma University, Japan;
(xxxiv) Harumichi Yoshiura, Tokyo Institute of Technology, Japan;
(xxxv) Takao Yuyama, Tokyo Institute of Technology, Japan;
(xxxvi) Mou Zhuoqun, RIMS - Kyoto University, Japan.

Benjamin Collas
RIMS - Kyoto University, Kyoto, Japan
Email: bcollas@kurims.kyoto-u.ac.jp
Office: RIMS room 302
http://www.kurims.kyoto-u.ac.jp/~bcollas/

Pierre Dèbes
Université de Lille, France
Email: Pierre.Debes@univ-lille.fr
Office: building M2, office 310 (Campus Lille 1)
https://math.univ-lille1.fr/~pde/